Primarily about Primaries

Allan Borodin  
University of Toronto

Omer Lev  
Ben-Gurion University

Nisarg Shah  
University of Toronto

Tyrone Strangway  
University of Toronto

Abstract

Much of the social choice literature examines direct voting systems, in which voters submit their ranked preferences over candidates and a voting rule picks a winner. Real-world elections and decision-making processes are often more complex and involve multiple stages. For instance, one popular voting system filters candidates through primaries: first, voters affiliated with each political party vote over candidates of their own party and the voting rule picks a candidate from each party, which then compete in a general election.

We present a model to analyze such multi-stage elections, and conduct the first quantitative comparison (to the best of our knowledge) of the direct and primary voting systems with two political parties in terms of the quality of the elected candidate. Our main result is that every voting rule is guaranteed to perform almost as well (i.e., within a constant factor) under the primary system as under the direct system. Surprisingly, the converse does not hold: we show settings in which there exist voting rules that perform significantly better under the primary system than under the direct system.

Introduction

If I could not go to heaven but with a party, I would not go there at all.

– Thomas Jefferson, 1789

Thomas Jefferson, like many of the US constitution’s authors, believed that political parties and factions are a bad thing (see also (Hamilton, Madison, and Jay 1787)). This view stemmed from a long history of British and English political history, in which prison sentences and executions were possible outcomes in the battle between factions for supremacy at the Royal court (Simms 2007). However, both in Britain and in the Unites States, once their respective legislative assemblies gained political force, parties turned out to be quite unavoidable. Even Jefferson had to start his own party, which ended up quite successful, and was able to vanquish the opposing party from political existence (Wilentz 2005).

Fast forwarding to today, political parties have become the bedrock of parliamentary politics throughout the world. In particular, one of political parties’ main roles – if not the most important (especially in presidential systems) – is to select the candidates which are voted for by the general public.

The mechanisms by which parties make this selection are varied, and they have evolved significantly throughout the past 150 years. But in the past few decades there has been a marked shift by parties throughout the world towards increasing the ability of individual party (or unaffiliated) members to influence the outcome, and in some cases, to be the only element to determine party candidates (Cross and Blais 2011). In particular, US parties have changed their election methods since the 1970s to focus the selection of presidential, congressional and state-wide candidates on popular support by party members via primaries (Cohen et al. 2008).

Despite this long and established role of parties in whittling down the candidate field in elections, the treatment of a parties’ role in elections within the multiagent systems community has been quite limited. While various candidate manipulation attacks have been investigated (e.g., Sybil attacks), and there is recent research into parties as a collection of similar minded candidates (e.g., in gerrymandering, across different districts), the role of parties in removing candidates has not been explored.

The focus of this paper is the primary voting system, in which each party’s electorate selects a winner from among the party’s candidates, and among these primary winners, an ultimate election winner is selected by the general public. Our goal is to compare this system to the direct voting system, in which all voters directly vote over all candidates.

Our Results

Our contribution is twofold. First, we formulate a model which allows a quantitative comparison of the two voting systems. Our model is a spatial model of voting in which voters and candidates lie in an underlying multi-dimensional space, and voter preferences are single-peaked. This allows us to compare each candidate’s social utility in terms of its total distance to the voters. We make the evaluation metric formal using the notion of distortion advocated by a recent line of research (Procaccia and Rosenschein 2006; Boutilier et al. 2015).

Second, we use this model to present a comparison of the direct and primary voting systems. In particular, we show that no voting rule performs much worse (by more than a constant factor) under the primary system than under the direct system. While the converse holds in some cases, we show settings in which it does not, and there exist voting
It is important to note that while we write of parties, voters and elections, this multi-step model applies to a variety of decision-making processes by agents. An organization selecting an “employee of the month” may ask each unit to nominate a single candidate, and then choose from amongst them. A city may ask its regional subdivisions to assess which roads require urgent fixing, and then the city council decides from these options where to invest its efforts. Fundamentally, in many cases where the potential number of options is huge, it is common to use subdivisions to cull the options and present only a few of them for discussion and vote. In such cases, our multi-stage model is apt.

**Related Work**

The analysis of regular, direct elections is long and varied, both in the social sciences and in AI (Brandt et al. 2016). In our particular setting, the voters are located in a metric space, with their preferences related to their distance from candidates. Such settings have been widely investigated in the social science literature since the work of Downs (1957), recently summarized by Schofield (2008). In particular, we focus on the idea of distortion in such elections, a topic broached by Procaccia and Rosenschein (2006) for voters with a utility function, but investigated in the context of voters in metric spaces in a series of papers (Anshelevich, Bhardwaj, and Postl 2015; Anshelevich and Postl 2017; Skowron and Elkind 2017) for most common voting rules. Feldman, Fiat, and Golomb (2016) explored such a setting for strategyproof mechanisms.

Discussing changes to the set of candidates has been mainly focused on two paths of research. Strategic candidates, investigation of which began with the work of Dutta, Jackson, and Le Breton (2001) – followed by Dutta, Jackson, and Le Breton (2002) discussing strategic candidacy in tournaments, and recently further explored by Brill and Conitzer (2015), Polukarov et al. (2015) and others – deals mainly with finding equilibria. The other is the addition and removal of candidates, as a form of control manipulation, which was studied by Bartholdi III, Tovey, and Trick (1992); see the summaries by Brandt et al. (2016) and Rothe (2015).

Investigating parties’ selection methods and their effect on the election has mostly been done in the social sciences. Kenig (2009) details the range of selection methods parties use, and there has been significant focus on more democratic methods for leader selection (Cross and Katz 2013), which seems to be a general trend in many Western countries (Cross and Blais 2011). There is also significant literature on particular party elections in various countries, such as Britain (Jobson and Wickham-Jones 2010), Belgium (Wauters 2010), Israel (Hazar 1997), and many others. Naturally, the most widely examined country is the US, in which political parties have been a fixture of political life since its early days (Wilentz 2005). The most recent extensive summary of research on it is due to Cohen et al. (2008), who try to explain how party power-brokers influence the party membership vote. Norpeth (2004) uses primary data to predict election results, and notably Sides et al. (2018) show that primary voters are very similar to “regular” voters.

**Model**

For \( k \in \mathbb{N} \), define \([k] = \{1, \ldots, k\}\). Let \( V = [n] \) denote a set of \( n \) voters, and \( A \) denote a set of \( m \) candidates. We assume that voters and candidates lie in an underlying metric space \( M = (S, d) \), where \( S \) is a set of points and \( d \) is a distance function satisfying the triangle inequality and symmetry. More precisely, there exists an embedding \( \rho : V \cup A \to S \) mapping each voter and candidate to a point in \( S \). For a set \( X \subseteq V \cup A \), we slightly abuse the notation and let \( \rho(X) = \{\rho(x) : x \in X\} \). Also, for \( x, x' \in V \cup A \), we often use \( d(x, x') \) instead of \( d(\rho(x), \rho(x')) \) for notational convenience.

In this work, we assume that voters and candidates additionally have an affiliation with a political party. Specifically, we study a setting with two parties\(^1\), denoted \( -1 \) and \( 1 \). The party affiliation function \( \pi : V \cup A \to \{-1, 1\} \) maps each voter and candidate to the party they are affiliated with. For \( p \in \{-1, 1\} \), let \( V_p = \pi^{-1}(p) \cap V \), \( A_p = \pi^{-1}(p) \cap A \), \( n_p = |V_p| \), and \( m_p = |A_p| \). We require \( n_p \geq 1 \) for each \( p \in \{-1, 1\} \). Our main result (Theorem 2) holds even if there are independent voters not affiliated with either party.

Collectively, an instance \( I = (V, A, M, \rho, \pi) \). Given \( I \), our goal is to find a candidate \( a \in A \) as the winner. The social cost of \( a \) is its total distance to the voters, denoted \( C^I(a) = \sum_{i \in V} d(i, a) \). For party \( p \in \{-1, 1\} \), let \( C^I_p(a) = \sum_{i \in V_p} d(i, a) \). Hence \( C^I(a) = C^I_{-1}(a) + C^I_1(a) \).

We also use for \( X \subseteq V \), \( C_X^I(a) = \sum_{i \in X} d(i, a) \). Given an instance \( I \), we would like to choose a candidate \( a_{\text{OPT}} \in \arg \min_{a \in A} C^I(a) \) that minimizes the social cost. We shall drop the instance from superscripts if it is clear from the context.

However, we do not observe the full instance. Specifically, we do not know the underlying metric \( M \) or the embedding function \( \rho \). Instead, each voter \( i \in N \) submits a \( a \) vote, which is a ranking (strict total order) \( \succ_i \) over the candidates in \( A \) by their distance to the voter. Specifically, for all \( i \in N \) and \( a, b \in A \), \( a \succ_i b \iff d(i, a) \leq d(i, b) \). The voter is allowed to break ties between equidistant candidates arbitrarily. The vote profile \( \vec{\succ}_I = (\succ_1, \ldots, \succ_n) \) is the collection of votes. Given an instance \( I \), its corresponding election \( E^I = (V, A, \vec{\succ}_I, \pi) \) contains all observable information.

In the families of instances that we consider, we fix the number of candidates \( m \) and let the number of voters \( n \) be arbitrarily large. This choice is justified because in many typical elections (e.g., political ones), voters significantly outnumber candidates. Let \( T^m_{n, \alpha} \) be the family of instances satisfying the following conditions.

- Each party has at least an \( \alpha \) fraction of the voters affiliated with it, i.e., \( n_p \geq \alpha \cdot n \) for each \( p \in \{-1, 1\} \). Note that \( \alpha \in [0, 0.5] ; \alpha = 0.5 \) is the strictest (exactly half of the voters are affiliated with each party), while \( \alpha = 0 \) imposes the model can be extended to multiple parties in a reasonably straightforward manner, but for simplicity’s sake, we shall focus on only 2 parties.
no conditions; in the latter case, we omit the superscript $\alpha$.  

- The number of candidates is at most $m$.  

In particular, we shall focus on a few cases of $M$:

$M = \star$ That is, all metric spaces.

$M = \mathbb{R}^k$ We shall take the metric as be $M = (\mathbb{R}^k, d)$, where $d$ is the standard Euclidean distance.

$M = \text{sep-}\mathbb{R}^k$ This means the embedding $\rho$ must be such that $\rho(V_{-1} \cup A_{-1})$ and $\rho(V_1 \cup A_1)$ are linearly separable.\footnote{Two sets of points are linearly separable if the interiors of their convex hulls are disjoint, or equivalently, if there exists a hyperplane that contains each set in a distinct closed halfspace.} In this case we shall take the metric to be $M = (\mathbb{R}^k, d)$ with $d$ as the standard Euclidean distance. In plain words, the voters and candidates affiliated with each party reside in a certain part of the metric space, separate from those affiliated with the other party. In a single dimension, this means there exists a threshold on the line such that voters and candidates affiliated with one party lie to the left of it, while those affiliated with the other party lie to the right. Note that this is the only choice of $M$ that restricts the embedding $\rho$ based on party affiliation $\pi$.

These families of instances are related by the following relation. For all $k$,

$$T^\alpha_{m, \text{sep-}\mathbb{R}^k} \subset T^\alpha_{m, \text{sep-}R^{k+1}} \subset T^\alpha_{m, \mathbb{R}^{k+1}} \subset T^\alpha_{m, \star}$$

### Voting Rules and Distortion

A voting rule $f$ takes an election as input, and returns a winning candidate from $A$. We say that the cost-approximation of $f$ on instance $I$ is

$$\phi(f, I) = \frac{C^I(f(I'))}{\min_{a \in A} C^I(a)}.$$

and given a family of instances $I$, the distortion of $f$ with respect to $I$ is

$$\phi_I(f) = \sup_{I' \in I} \phi(f, I').$$

Since distortion is a worst-case notion, we have that $I \subseteq I'$, $\phi_I(f) \leq \phi_{I'}(f)$ for every voting rule $f$.

Standard voting rules choose the winning candidate independently of party affiliations. These include rules such as plurality, Borda count, $k$-approval, veto, and STV. We refer readers to the book by Brandt et al. (2016) for their definitions. We call a voting rule affiliation-independent if $f(E) = f(E')$ when elections $E$ and $E'$ differ only in their party affiliation functions. Since an affiliation-independent voting rule $f$ ignores party affiliations, we have $\phi_{T^\alpha_{m, \text{sep-}R^k}}(f) = \phi_{T^\alpha_{m, \star}}(f)$. All of the above-mentioned rules, in addition to being affiliation-independent, share the property of being unanimous, i.e., they return candidate $a$ when $a$ is the top choice of all voters.

### Stages and Primaries

Given an affiliation-independent voting rule $f$, voting systems with primaries employ a specific process to choose the winner, essentially resulting in a different voting rule $\hat{f}$ that operates on a given election $E = (V, A, \succ, \pi)$ as follows:

1. First, it creates two primary elections: for $p \in \{-1, 1\}$, define $E_p = (V_p, A_p, \succ_p, \pi_p)$, where $\succ_p$ denotes the preferences of voters in $V_p$ over candidates in $A_p$, and $\pi_p : V_p \to \{p\}$ is a constant function.

2. Next, it computes the winning candidate in each primary election (primary winner) using rule $f$: for $p \in \{-1, 1\}$, let $a_p^\ast = f(E_p)$.

3. Finally, let $E_g = (V, \{a_{-1}^\ast, a_1^\ast\}, \succ_g, \pi)$ be the general election, where $\succ_g$ denotes the preferences of all voters over the two primary winners. The winning candidate is $\hat{f}(E) = \text{maj}(E_g)$.

This resembles systems employed by the main US, Canadian and other countries’ parties, in which a party’s members vote on their party’s candidates to select a winner of their primary. In other systems, such selection could be a multi-stage process.

Given an affiliation-independent voting rule $f$, the goal of this paper is to compare its performance under the direct system, in which $f$ is applied on the given election directly, to its performance under the primary system, in which $f$ is applied on the given election instead. Formally, given a family of instances $I$ and an affiliation-independent voting rule $f$, we wish to compare $\phi_I(f)$ and $\phi_{\hat{f}}(f)$ (henceforth, the distortion of $f$ with respect to $I$ under the direct and the primary systems, respectively).

### Small Primaries Are Terrible

Recall that in a family of instances $T^\alpha_{m, M}$, we require that at least $\alpha$ fraction of voters be affiliated with each party, i.e., $n_p \geq \alpha n$ for each $p \in \{-1, 1\}$. In other words, each primary election must have at least $\alpha n$ voters.

We first show that when a primary election can have very few voters ($\alpha = 0$), every reasonable voting rule has an unbounded distortion in the primary system, even with respect to our most stringent family of instances $T^\alpha_{m, \text{sep-}R}$.

**Theorem 1.** For $m \geq 3$, $\phi_{T^\alpha_{m, \text{sep-}R}}(\hat{f}) = \infty$ for every affiliation-independent unanimous voting rule $f$.

**Proof.** Consider an instance $I \in T^\alpha_{m, \text{sep-}R}$ in which voter $1$ is located at $0$ and affiliated with party $-1$, while the remaining $n - 1$ voters are located at $1$ and affiliated with party $1$. All $m$ candidates are affiliated with party $-1$; one is at $0$, and the rest are at $1$.

\footnote{If one of the parties has no affiliated candidates, then the primary winner of the other party becomes the overall winner. In a setting with more than 2 parties, or where each party nominates several candidates, the general election can use $f$ to determine the outcome (or use some other voting process).}
The candidate \( a^* \) at 0 becomes the primary winner of party \(-1\), and trivially becomes the overall winner. Its social cost is \( C(a^*) = n - 1 \). In contrast, an optimal candidate \( a_{OPT} \) at 1 has social cost \( C(a_{OPT}) = 1 \). Hence, \( \frac{C(a^*)}{C(a_{OPT})} = n - 1 \). Since the number of voters is \( n \) is unbounded, \( \phi_{\text{sub}}(\hat{f}) = \infty \).

Theorem 1 continues to hold even if we require that at least a constant fraction of candidates be affiliated with each party: we could simply move a constant fraction of the candidates from 1 to 3, and the proof would still go through.

On the other hand, if we require that at least a constant fraction of voters be affiliated with each party, the result changes dramatically.

**Large Primaries Are Never Much Worse**

For every affiliation-independent voting rule \( f \), we bound the distortion of \( \hat{f} \) in terms of the distortion of \( f \) for every instance. Note that this is stronger than comparing the worst-case distortions of \( f \) and \( \hat{f} \) over a family of instances.

Given an instance \( I = (V, A, M, \rho, \pi) \) and party \( p \in \{-1, 1\} \), we say that \( I_p = (V_p, A_p, M, \rho_p, \pi_p) \) is the primary instance of party \( p \), where \( \rho_p \) and \( \pi_p \) are restrictions of \( \rho \) and \( \pi \) to \( V_p \cup A_p \). The primary election \( E_p \) of party \( p \) is precisely the election corresponding to instance \( I_p \).

**Theorem 2.** Let \( I = (V, A, M, \rho, \pi) \) be an instance. For \( p \in \{-1, 1\} \), let \( I_p \) be the primary instance of party \( p \), and \( n_p = |V_p| \geq \alpha n \). Then,

\[
\phi(\hat{f}, I) \leq 3 \cdot \frac{1 - \alpha + \max(\phi(f, I_{-1}), \phi(f, I_1))}{\alpha}.
\]

Further, for a socially optimal candidate \( a_{OPT} \in \arg \min_{a \in \Lambda} C^I(a) \), we have a bound depending only on the distortion of the primary instance of its party:

\[
\phi(\hat{f}, I) \leq 3 \cdot \frac{1 - \alpha + \phi(f, I_{a_{OPT}})}{\alpha}.
\]

For each family of instances \( \mathcal{I} \) that we study, it holds that for every instance \( I \in \mathcal{I} \), both its primary instances, if seen as direct elections, are also in \( \mathcal{I} \) (since the party division has no effect on the direct election distortion). Hence, we can convert the instance-wise comparison to a worst-case comparison.

**Corollary 3.** For every \( \alpha > 0 \), \( k \in \mathbb{N} \), family of instances \( \mathcal{I} \in \{I_{m, s}, I_{m, \text{sep}}, I_{m, \text{sep}}\} \), and affiliation-independent voting rule \( f \),

\[
\phi_{\mathcal{I}}(\hat{f}) \leq 3 \cdot \frac{1 - \alpha + \phi_{\mathcal{I}}(f)}{\alpha}.
\]

Note that \( \phi_{\mathcal{I}}(f) \geq 1 \) by definition. Hence, we can write \( \phi_{\mathcal{I}}(\hat{f}) \leq \frac{6}{\alpha} \cdot \phi_{\mathcal{I}}(f) \). In other words, for every affiliation-independent voting rule \( f \), its distortion under the primary system is at most a constant times bigger than its distortion under the direct system, with respect to every family of instances that we consider.

To prove Theorem 2, we need two lemmas. The first lemma shows that if the distortion of a rule \( f \) for a primary instance \( I_p \) is low, then the corresponding primary winner \( a^*_p \) is nearly as good as any candidate in \( A_p \) for the overall election as well. The intuition is that when voters in \( V \setminus V_p \) drive up the social cost of \( a^*_p \) (i.e., when they are far from \( a^* \)), they must do so for every candidate in \( A_p \).

**Lemma 4.** Let \( a^*_p \) denote the primary winner of party \( p \). Then

\[
C(a^*_p) \leq 1 - \alpha + \frac{\phi(f, I_p)}{\alpha} \cdot \min_{a \in A_p} C(a).
\]

**Proof.** Let \( \theta = \phi(f, I_p) \) (hence \( \theta \geq 1 \)). Fix an arbitrary \( a \in A_p \). Then, \( C_{V_p}(a^*_p) \leq \theta \cdot C_{V_p}(a) \). Now,

\[
C(a^*_p) = C_{V_p}(a^*_p) + C_{V \setminus V_p}(a^*_p)
\]

\[
\leq \theta \cdot C_{V_p}(a) + C_{V \setminus V_p}(a) + (n - n_p) \cdot d(a, a^*_p)
\]

\[
\leq \theta \cdot C(a) + (n - n_p) \cdot d(a^*_p, a),
\]

where the second transition follows due to the triangle inequality. We also have \( d(a^*_p, a) \leq d(a^*_p, i) + d(i, a) \) for any \( i \in V_p \). Thus,

\[
d(a^*_p, a) \leq C_{V_p}(a^*_p) + C_{V \setminus V_p}(a^*_p)
\]

\[
\leq \frac{1 + \theta}{n_p} \cdot C_V(a) \leq \frac{1 + \theta}{n_p} \cdot C(a).
\]

Substituting Equation (2) into Equation (1), and using the fact that \( \frac{n - n_p}{n_p} \leq \frac{1 - \alpha}{\alpha} \), we get the desired result.

Our next lemma shows that the primary winner that wins the general election is not much worse than the primary winner that loses the general election.

**Lemma 5.** Let \( a^*_1 \) and \( a^*_1 \) be the two primary winners, \( a^* \in \{a^*_1, a^*_1\} \) be the winner of the general election, and \( \tilde{a} \in \{a^*_1, a^*_1\} \setminus \{a^*\} \). Then,

\[
C(a^*) \leq 3 \cdot C(\tilde{a}).
\]

**Proof.** Because \( a^* \) wins the general election by a majority vote, there must exist \( X \subseteq V \), \( |X| \geq \frac{n}{2} \) such that \( d(i, a^*) \leq d(i, \tilde{a}) \) for every \( i \in X \). Combining with the triangle inequality \( d(a^*, \tilde{a}) \leq d(a^*, i) + d(i, \tilde{a}) \), we get

\[
d(i, \tilde{a}) \geq d(i, \tilde{a}) - \frac{d(a^*, \tilde{a})}{2} \text{ for every } i \in X.
\]

\[
C(\tilde{a}) \geq \frac{n}{2} \cdot d(a^*, \tilde{a}) \Rightarrow d(a^*, \tilde{a}) \leq \frac{4}{n} \cdot C(\tilde{a}.
\]

Now, we have

\[
C(a^*) = \sum_{i \in X} d(i, a^*) + \sum_{i \in X} d(i, a^*)
\]

\[
\leq \sum_{i \in X} d(i, \tilde{a}) + \sum_{i \in X} d(i, \tilde{a}) + d(a^*, \tilde{a})
\]

\[
\leq C(\tilde{a}) + \frac{n}{2} \cdot d(a^*, \tilde{a}),
\]

where the second transition follows because \( d(i, a^*) \leq d(i, \tilde{a}) \) for \( i \in X \), and \( d(i, a^*) \leq d(i, \tilde{a}) + d(a^*, \tilde{a}) \) due to the triangle inequality. Substituting Equation (3) into Equation (4), we get the desired result.
We are now ready to prove our main result.

**Proof of Theorem 2.** Recall that $a^*_{\alpha,1}$ and $a^*_1$ are the primary winners. Let $p \in \{-1, 1\}$ be such that $a^*_p$ is the winner of the general election. Let $a_{OPT} \in \arg \min_{a \in A} C(a)$ be a socially optimal candidate. We consider three cases.

- **Case 1:** $a_{OPT} \in A_p$. That is, the optimal candidate and the winner are affiliated with the same party. In this case, Lemma 4 yields
  \[
  \phi(f, I) \leq 1 - \alpha + \phi(f, I_p) = 1 - \alpha + \phi(f, I_{\pi(a_{OPT})}).
  \]

- **Case 2:** $a_{OPT} \in A_p$, $a_{OPT} \neq a^*_p$. That is, the optimal candidate is affiliated with a party different than that of the winner, and is not a primary winner. In this case, we use both Lemmas 4 and 5 to derive
  \[
  \phi(f, I) = \frac{C(a^*_p)}{C(a_{OPT})} \cdot \frac{C(a^*_p)}{C(a^*_p)} \cdot \frac{C(a^*_p)}{C(a_{OPT})} \\
  \leq 3 \cdot \frac{1 - \alpha + \phi(f, I_p)}{\alpha} \\
  = 3 \cdot \frac{1 - \alpha + \phi(f, I_{\pi(a_{OPT})})}{\alpha}.
  \]

Thus, in each case, we have the desired approximation.

Note that our proof of Theorem 2 does not preclude the existence of independent voters. Specifically, we can allow the party affiliation function $\pi$ to map a subset of voters $V_0$ to a neutral choice (say 0), have these voters only vote in the general election and not in either primary election under the primary system, and the proof of Theorem 2 would still go through. Additionally, we can also relax the restriction that all new voters are affiliated in the general election. That is, we can assume that a subset of voters $V_0 \subseteq V$ vote in the general election under the primary system, assume $|V_0| \geq \gamma n$, and a version of Theorem 2 would still hold in which the constant 3 in the bound on the distortion is replaced by $\frac{4}{\gamma}$. This requires a generalization of Lemma 5, which is presented as Theorem 10 in the appendix.

**Large Primaries Are Not Better Without Party Separability**

While we showed in the previous section that a voting rule does not perform much worse under the primary system than under the direct system, we now show that it does not perform any better either, at least in the worst case over all instances with at most $m$ candidates. The result continues to hold even if we require each party to have at least a constant fraction of the voters.

Note that this result is weaker than Theorem 2 because it is a worst-case comparison instead of an instance-wise comparison. However, it still applies to all voting rules $f$. It applies to any metric that does not require separability of parties, in particular to $\mathcal{I}_{m, *}$ and $\mathcal{I}_{m, R^k}$.

**Theorem 6.** For every $k \in \mathbb{N}$, $\alpha \in [0, 0.5]$, and $\mathcal{M}$ a metric which does not require party separability, and affiliation-independent voting rule $f$, we have $\phi_{\mathcal{I}_{m, \mathcal{M}}}(f) \geq \phi_{\mathcal{I}_{m, \alpha}}(f)$.

**Proof.** We shall denote $\mathcal{I}_{m, \mathcal{M}}^\alpha$ as $\mathcal{I}$. We want to show that for every instance $I \in \mathcal{I}$, there exists an instance $I' \in \mathcal{I}$ such that $\phi(f, I') \geq \phi(f, I)$. This would imply the desired result.

Fix an instance $I = (V, A, M, \rho, \pi) \in \mathcal{I}$. Let $a_{OPT} \in A$ denote an optimal candidate in $I$, and $a^* = f(E^4)$. Note that $\phi(f, I) = \frac{C^4(a^*)}{C^4(a_{OPT})}$. Construct instance $I' = (V', A, M, \rho', \pi')$ as follows.

- Let $V' = V \cup \tilde{V}$, where $\tilde{V}$ is a new set of voters and $|\tilde{V}| = |V|$.
- Let $\rho'(x) = \rho(x)$ for all $x \in V \cup A$, and $\rho'(x) = \rho(a_{OPT})$ for all $x \in \tilde{V}$. That is, $\rho'$ matches $\rho$ for existing voters and candidates, and the new voters are co-located with $a_{OPT}$.
- Let $\pi'(x) = -1$ for all $x \in V \cup A$, and $\pi'(x) = 1$ for all $x \in \tilde{V}$. That is, all existing voters and candidates are affiliated with party $-1$, while all new voters are affiliated with party 1.

First, let us check that $I' \in \mathcal{I}$. Since $I$ has $m$ candidates, so does $I'$. Further, in $I'$, we have $|V'_{\downarrow 1}| = |V'_1| = |V'|/2$, which satisfies the constraint corresponding to every $\alpha \in [0, 0.5]$. Hence, we have $I' \in \mathcal{I}$.

Let us apply $\hat{f}$ on $I'$. One of its primary instances, $I'_{\downarrow 1}$, is precisely $I$. Hence, the primary winner of party $-1$ is $f(I'_{\downarrow 1}) = f(I) = a^*$. Because there are no candidates affiliated with party 1, $a^*$ becomes the overall winner.\(^4\)

Next, $C^4(a^*) \geq C^4(a^*)$ because $V \subseteq V'$. Also, $C^4(a_{OPT}) = C^4(a_{OPT})$ because $a_{OPT}$ has zero distance to all voters in $V' \setminus V$. Together, they yield

\[
\phi(\hat{f}, I') = \frac{C^4(a^*)}{C^4(a_{OPT})} \geq \frac{C^4(a^*)}{C^4(a_{OPT})} = \phi(f, I),
\]

as desired.

\(\Box\)

Our proof actually establishes a slightly stronger result. Instead of showing $\phi_{\mathcal{I}_{m, \mathcal{M}}}(f) \geq \phi_{\mathcal{I}_{m, \alpha}}(f)$, we actually show $\phi_{\mathcal{I}_{m+1, \mathcal{M}}}(f) \geq \phi_{\mathcal{I}_{m, \mathcal{M}}}(f)$.

\(^4\)Even if we require each party to have at least one affiliated candidate, the proof essentially continues to hold. In this case, we can add one candidate affiliated with party 1 that is located sufficiently far from all the voters, ensuring that $a^*$ still becomes the overall winner. This would show $\phi_{\mathcal{I}_{m+1, \mathcal{M}}}(f) \geq \phi_{\mathcal{I}_{m, \mathcal{M}}}(f)$ because instance $I'$ may now have $m + 1$ candidates.
Separability and Its Advantages

The analysis for $\mathcal{I}_{m,sep-R}$ is not so straightforward. In the proof of Theorem 6, we co-located the new voters affiliated with party 1 and $a_{OPT}$ affiliated with party $-1$. This was allowed because non-separable metrics like $\star$ and $\mathbb{R}^k$ place no constraints on the embedding.

With $\mathcal{I}_{m,sep-R}$, we need the voters and candidates affiliated with one party to be separated from those affiliated with the other. Hence, this operation of putting all of one party’s voters at the location of $a_{OPT}$ belonging to another party would be allowed only if, in the original instance $I$, $a_{OPT}$ is on the boundary of the convex hull of $\rho(V \cup A)$. While this is not the case for all instances, we only need this in at least one worst-case instance for $f$, i.e., for at least one $I \in \mathcal{I}_{m,sep-R}$ with $\phi(f, I) = \phi_{\mathcal{I}_{m,sep-R}}(f)$. Equation (5) would then yield the desired result. More generally, it is sufficient if, given any $\epsilon > 0$, we can find an instance $I$ such that $\phi(f, I) \geq \phi_{\mathcal{I}_{m,sep-R}}(f) - \epsilon$ and $a_{OPT}$ is at distance at most $\epsilon$ from the boundary of the convex hull of $\rho(V \cup A)$.

Interestingly, (Anshelevich, Bhardwaj, and Postl 2015) show that this is indeed the case for plurality and Borda count (see the proof of their Theorem 4). Thus, we have the following.

Proposition 7. Let $f$ be plurality or Borda count. Then, $\phi_{\mathcal{I}_{m,sep-R}}(f) \geq \phi_{\mathcal{I}_{m,sep-R}}(f)$.

However, known worst cases for the Copeland rule (Anshelevich, Bhardwaj, and Postl 2015) and STV (Skowron and Elkind 2017) do not satisfy this requirement. It is unknown if these rules admit a different worst case that satisfies it.

This raises an important question. Does Proposition 7 hold for all affiliation-independent voting rules? We shall shortly answer this negatively.

More precisely, we construct an affiliation-independent voting rule $f$ such that $\phi_{\mathcal{I}_{m,sep-R}}(f) \ll \phi_{\mathcal{I}_{m,sep-R}}(f)$ for every $\alpha > 0$. That is, with large primaries, $f$ performs much better under the primary system than under the direct system, when voters and candidates are embedded on a line and the separability condition is imposed.

Note that instances in $\mathcal{I}_{m,sep-R}$ are highly structured. For instance, it is known that when voters and candidates are embedded on a line, there always exists a weak Condorcet winner (Black 1948), and selecting such a candidate results in a distortion of at most 3 (Anshelevich, Bhardwaj, and Postl 2015). Hence, we have $\phi_{\mathcal{I}_{m,sep-R}}(f) = 3$ for every Condorcet-consistent, affiliation-independent voting rule $f$.

Our aim in this section is to construct an affiliation-independent voting rule $f_{fail}$ that with respect to $\mathcal{I}_{m,sep-R}$ has an unbounded distortion in the direct system, but at most a constant distortion in the primary system.

Definition 1. Let $f_{fail}$ be an affiliation-independent voting rule that operates on election $E = (V, A, \succ)$ as follows. Let $A = \{a_1, \ldots, a_m\}$, and $t = (m + 1)/2$.

- **Special Case:** If $m \geq 9$, $m$ is odd, $n \geq m^2$, and $\nabla$ has the following structure, then return $a_1$.
  1. For voter 1, $a_1 \succ \ldots \succ a_m$.
  2. For voter 2, $a_m \succ \ldots \succ a_2$.
  3. For voter 3, $a_{t-1}$ is the most preferred, and $a_{m-2} \succ a_1 \succ a_3 \succ a_{m-1} \succ a_m$.
  4. For voter 4, $a_{t+1}$ is the most preferred, and $a_3 \succ a_m \succ a_4 \succ a_2$.
  5. For $j \in [m - 2]$, for voter $i = 4 + (2j - 1)$, $a_{j+1} \succ a_j \succ a_{j+2} \succ a_j$, and for voter $i' = 4 + 2j$, $a_{j+1} \succ a_{j'} \succ a_j \succ a_{j+2}$.
  6. For every other voter $v$, $a_t$ is the most preferred.

- If $E$ does not fall under the special case, then apply any Condorcet consistent voting rule (e.g., Copeland’s rule).

Note that $m$ being odd ensures that $t$ is an integer, and $m \geq 9$ ensures that $a_1, a_3, a_{t-1}, a_t, a_{t+1}, a_{m-2}$, and $a_m$ are all distinct candidates. The significance of $n \geq m^2$ will be clear later.

We will now establish that a worst-case instance of $f_{fail}$ falls under the special case; for this instance, we need to show that $a_t$ is socially optimal; that $f_{fail}$ returns $a_1$ on this instance; and most importantly, that the structure of $\nabla$ ensures that the optimal candidate $a_t$ is sufficiently far from both the leftmost and the rightmost candidates.

We prove this last fact in the following lemma.

Lemma 8. Let $I = (V, A, M, \rho, \pi) \in \mathcal{I}_{m,sep-R}$ be an instance for which the corresponding election $E_i$ falls under the special case of $f_{fail}$. Then the following holds.

1. Either $\rho(a_1) \leq \ldots \leq \rho(a_m)$, or $\rho(a_1) \geq \ldots \geq \rho(a_m)$, or $\rho(A) = 2$.
2. If $|\rho(A)| \neq 2$, $\min \{d(a_i, a_j), d(a_i, a_m)\} \geq d(a_1, a_m)/4$.

The proof of the lemma, and the proof of the next theorem is in the appendix due to space constraints.

Theorem 9. For $m \geq 9$ and constant $\alpha \in (0, 0.5]$, $\phi_{\mathcal{I}_{m,sep-R}}(f_{fail})$ is upper bounded by a constant, whereas $\phi_{\mathcal{I}_{m,sep-R}}(f_{fail})$ is unbounded.

Using Simulations to Go Beyond Worst Case

So far we compared the distortion of a voting rule under the direct and primary systems, taken in the worst case over a family of instances. In practice, such worst case instances may not arise naturally. In this section, we compare the distortion of a voting rule under the direct and primary systems, in the average case over simulated instances. To control for the effect of vastly different numbers of voters or candidates affiliated with the two parties, we place the restriction that half of the voters and half of the candidates must be affiliated with each party. Furthermore, we wished to examine instances in which the voters and candidates affiliated with one party lie in a separate region in the metric space than those affiliated with the other party.

We fix the space to be $[0, 1]^k$ for $k \in \{1, 3, 5, 7, 9\}$. We generate 1000 instances satisfying the following restrictions. First, we place a set $V$ of $n = 1000$ voters at uniformly
random locations in $[0,1]^k$. Next, we find a threshold $t$ such that the locations of half of the voters (call this set $V_{-1}$) are on one side of the threshold, while the locations of the rest (call this set $V_1$) are on the other. We do not affiliate the voters yet. Next, we place $\frac{m}{2} = 10$ candidates (call this set $A_{-1}$) uniformly at random on one side of the threshold, and $\frac{m}{2} = 10$ candidates (call this set $A_1$) uniformly at random in the other.

Once the locations of the voters and candidates are fixed, we create two instances. In one instance (SPLIT), we assign $V_{-1} \cup A_{-1}$ to party $-1$, and assign $V_1 \cup A_1$ to party 1. This instance belongs to $T_{m,\text{sep},R^k}^{0.5}$. In the other instance (RANDOM), we assign half of the voters and half of the candidates – chosen uniformly at random – to party $-1$, and the rest to party 1. This instance belongs to $T_{m,R^k}^{0.5}$, but not necessarily to $T_{m,\text{sep},R^k}^{0.5}$. This allows us to directly compare the effect party affiliation has on the distortion. Finally, we run five voting rules – plurality, Borda, STV, Copeland and maximin – on both instances under the direct and primary systems, and measure the distortion. Note that their distortion under the direct system would be identical for both instances because the two instances only differ in party affiliations. Thus, for each rule, we obtain three numbers: direct, primary-split, and primary-random. We average the distortion numbers across the 1000 instances.

See Figure 1 for selected simulation results. Higher dimension results resemble $k = 3$, only with decreasing values on the $y$ axis, as overall distortion levels dropped, but the comparison between primary and direct outcomes remained broadly the same. For each group of three election settings we ran a repeated measures ANOVA comparing the distortion values, and in all but 2 of 25 cases was had a $p$ value of under 0.05 (those were when $k = 9$). Perhaps the most important observation is that our simulation results stand in direct contrast to our worst case bounds. In almost all of our settings the distortion of a primary election (SPLIT or RANDOM) was better than that of its direct counterpart, and often it showed a significant improvement. This is especially noticeable in the non-Condorcet consistent rules (Plurality, Borda and STV), as in all but 3 of the 15 cases the distortion was significantly improved upon by using either primary, and this was most pronounced with plurality. With Condorcet consistent rules the distortion values were very low, regardless if a primary was used or not. In general, as we increase the dimension the groups become more homogeneous and the $p$ values grow while all of the distortions approach 1.

**Discussion**

Our paper initiates the novel quantitative study of multi-stage elections (and their comparison to single-stage elections), but leaves plenty to explore. Some directions are fairly straightforward extensions of our results. The most straightforward question is to tighten our bounds. There is also the question of explaining the trends we observe in the average case, which sometimes differ from our worst-case results. A next step would be to study realistic distributional models of voter preferences and candidate positions in the political spectrum, and analyze their effect on distortion.

Other extensions are seemingly more involved. Extending our framework to more than two parties requires the use of a ranked voting rule in the general election, which may significantly affect the analysis. Interestingly, such an extension would also incorporate independent candidates because one can imagine an independent candidate to be a party of their own. Examining the use of multiple and different voting rules as Narodytska and Walsh (2013) do for two-step voting (though without candidate elimination between stages) is an enticing direction. For example, in a multi-party direct system, we may use plurality, whereas in the primary system, the parties may use STV. It is also reasonable to consider that each party has its own voting rule. It would be interesting as well to examine party manipulation techniques in primary systems. Similarly, it is reasonable to believe that candidates may also strategically shift, to some extent, their location following the primaries, to make themselves more appealing to the general electorate.

We believe that the study of multi-stage elections and party mechanisms can not only contribute novel theoretical challenges to tackle, but can also bring research on computational social choice closer to reality and increase its impact.
References


Proof of Lemma 8

Proof. Since voter 1 ranks $a_m$ last, and preferences are single peaked on the line, $a_m$ is at one edge of the candidate ordering. Similarly, since voter 2 ranks $a_1$ last, candidate $a_1$ is also at the edge of the candidate ordering (i.e., $\rho(a_1) = \text{max}_{a \in A} \rho(a)$ or $\rho(a_1) = \text{min}_{a \in A} \rho(a)$ and $\rho(a_m) = \text{max}_{a \in A} \rho(a)$ or $\rho(a_m) = \text{min}_{a \in A} \rho(a)$). If $\rho(a_1) = \rho(a_m)$, this means voters 1 and 2 are located in an equal distance from all candidates (which means all candidates are located in the same location, or some are at some distance from voters 1 and 2, and the rest are at the same distance in the other direction from these voters).

Assume $|\rho(A)| > 2$ (this also means $\rho(a_1) \neq \rho(a_m)$ and $\rho(v_1) \neq \rho(v_2)$), we wish to show the order of candidates is as voter 1 ordered them, i.e., $\rho(a_1) \leq \ldots \leq \rho(a_m)$ or $\rho(a_1) \geq \ldots \geq \rho(a_m)$. If voter 1 is further away from all candidates (i.e., if $\rho(a_1) = \text{max}_{a \in A} \rho(a)$, $\rho(v_1) > \rho(a_1)$, and if $\rho(a_1) = \text{min}_{a \in A} \rho(a)$, $\rho(v_1) < \rho(a_1)$), the ordering of the candidates is as voter 1 orders them. Otherwise, let $t$ be the smallest index such that $\rho(a_1) \neq \rho(a_t)$, then $\rho(v_1)$ may be between $\rho(a_1)$ and $\rho(a_t)$. If $d(v_1, a_1) < d(v_1, a_t)$, once again, the ordering of candidates is as voter 1 ordered them. If $d(v_1, a_1) = d(v_1, a_t)$, for any $t' > t$, $\rho(a_{t'}) < \rho(a_1)$, as that contradicts voter 2’s vote ($a_t >_2 a_1$). Therefore, $\rho(t')$ is either at $\rho(t)$, or further away from $a_1$, meaning that candidates locations are ordered in the order voter 1 ordered them.

For the second condition, we will show that $d(a_1, a_t) \geq d(a_1, a_m)/4$. By symmetry, we also obtain $d(a_1, a_m) \geq d(a_1, a_t)/4$. This is trivial if $|\rho(A)| = 2$ and $\rho(a_1) = \rho(a_m)$. Now let us assume $|\rho(A)| \neq 2$. Without loss of generality, let $\rho(a_1) \leq \ldots \leq \rho(a_m)$ from the first condition. We show that either $\rho(a_1) = \ldots = \rho(a_m)$ or $\rho(a_1) < \ldots < \rho(a_m)$. If not, then we can find three consecutive candidates $a_j, a_{j+1}$, and $a_{j+2}$ such that either $\rho(a_j) = \rho(a_{j+2}) < \rho(a_j+1)$ or $\rho(a_j) < \rho(a_{j+2}) < \rho(a_j+1)$. Both possibilities are forbidden due to the existence of voters with preferences $a_{j+1} > a_j > a_{j+2}$ and $a_{j+2} > a_{j+1} > a_j$.

Now, if $\rho(a_1) = \ldots = \rho(a_m)$, then the second condition is trivially true. Suppose $\rho(a_1) < \ldots < \rho(a_m)$. Because voter 3 (resp. 4) prefers candidate $a_{t-1}$ (resp. $a_{t+1}$) the most, we have $\rho(3) \in (\rho(a_{t-1}), \rho(a_t))$ (resp. $\rho(4) \in (\rho(a_t), \rho(a_{t+1}))$. We can now show
\[
d(a_1, a_m) \leq d(a_1, v_4) + d(v_4, a_m) \\
\leq 2d(a_1, v_4) \quad (\because a_m >_4 a_1) \\
\leq 2d(a_1, a_{m-2}) \quad (\because \rho(a_t) < \rho(4) < \rho(a_{m-2})) \\
\leq 2d(a_1, v_3) + d(v_3, a_{m-2}) \\
\leq 4d(a_1, v_3) \quad (\because a_{m-2} >_3 a_1) \\
\leq 4d(a_1, a_4). \quad (\because \rho(a_1) < \rho(3) < \rho(a_t))
\]

This concludes the proof. \qed

Proof of Theorem 9

Proof. First, we show that $\phi_{T_m, \text{sep}}(f_{\text{fail}})$ is unbounded. Consider the following instance $I = (V, A, M, \rho, \pi)$. Let $V = \{v_1, \ldots, v_{2n}\}$ (where $n \geq m^2$), $A = \{a_1, \ldots, a_m\}$ ($m$ being odd), and $M = (\mathbb{R}, d)$ with $d$ being the Euclidean distance on the line.

The embedding function $\rho$ is as follows. For $\ell \in [m]$, $\rho(a_\ell) = \frac{\ell-1}{m-1}$; that is, candidates $a_1$ through $a_m$ are uniformly spaced in $[0, 1]$ with $\rho(a_1) = 0$ and $\rho(a_m) = 1$.

Fix $\varepsilon < 1/m^2$. The voters are embedded as follows.
\[
\rho(1) = \rho(a_1) - \varepsilon, \\
\rho(2) = \rho(a_m) + \varepsilon, \\
\rho(3) = \rho(a_1 - 1) + \varepsilon, \\
\rho(4) = \rho(a_1 + 1) - \varepsilon, \\
\rho(4 + (2j - 1)) = \rho(a_{j+1}) + \varepsilon \quad \forall j \in [m - 2], \\
\rho(4 + 2j) = \rho(a_{j+1}) - \varepsilon \quad \forall j \in [m - 2], \\
\rho(j) = \frac{1}{2} \quad \forall j \geq 2m + 1.
\]

Finally, in the party affiliation $\pi$, since we are just showing how bad the distortion of direct elections are and $f$ is affiliation-independent, the party affiliation is not important, and the outcome is independent of $\pi$, so we can assign it in any way such that half the voters are of one party and half are of the other. This construction simply ensures (and it is easy to check) that $I \in T_{m, \text{sep}}^0 \subseteq T_{m, \text{sep}}^\alpha$ for all $\alpha \in (0, 0.5)$.

Next, it is also easy to check that election $E^I$ falls under the special case of $f_{\text{fail}}$. Hence, $f_{\text{fail}}(E^I) = a_1$. Note that $C(a_1) \geq (2n - 2m) \cdot |\frac{1}{4} - \frac{1}{2}| > n - m$ because $a_1$ is at distance $\frac{1}{2}$ from all but 2m voters located at $\frac{1}{4}$. In contrast, $C(a_1) \leq 2m \cdot 1$ because $a_1$ is at zero distance from all but 2m voters (and its distance from those 2m voters is at most 1). Thus, $\phi(f_{\text{fail}}, I) \geq (n - m)/2m$. Since $n$ is unbounded, $\phi_{T_m, \text{sep}}(f_{\text{fail}})$ is also unbounded.

Finally, we show that $\phi_{T_m, \text{sep}}(\tilde{f}_{\text{fail}})$ is upper bounded by a constant. Fix an instance $I \in T_{m, \text{sep}}^\alpha$. For notational simplicity, we refer to the number of candidates in $I$ as $m$, though the proof below works if it is less than $m$. First, assume $|\rho(A)| \neq 2$. Without loss of generality, assume that $\rho(a_1) \leq \ldots \leq \rho(a_m)$, and that for a fixed $q \in [m]$, candidates $a_1, \ldots, a_q$ are affiliated with party $-1$ and the rest are affiliated with party 1.

Let $I_1$ and $I_2$ be the primary instances corresponding to $I$. Let $a_{OPT}$ be an optimal candidate for $I$. Without loss of generality, suppose it is affiliated with party $-1$.

In the proof of Theorem 2, $\phi(f, I)$ depends only on the distortion of $f$ on the primary instance of the party that $a_{OPT}$ is affiliated with. Hence, if primary election $E^I$ does not fall under the special case of $f_{\text{fail}}$, then $f_{\text{fail}}$ applies a Condorcet-consistent rule on $I_1$, ensuring that $\phi(f, I_1)$ is at most 3. In this case, by Theorem 2, $\phi(f, I)$ is also upper bounded by a constant.

Suppose $E^{I - 1}$ falls under the special case of $f_{\text{fail}}$. Let $t = (q + 1)/2$ and $d_{-1} = d(a_1, a_q)$. Then, by Lemma 8, $\min\{d(a_1, a_2), d(a_1, a_q)\} \geq d_{-1}/4$. Hereinafter, we use asymptotic notation liberally for simplicity.

Recall that there is a set of voters $S \subseteq V_1$ whose top candidate was $a_1$, and $|S| = |V_1| - 2q = \Omega(|V_1|) = \Omega(n)$,
where the second transition holds because in the special case, $|V_{-1}| \geq q^2$, and the final transition holds because $|V_{-1}| \geq \alpha n$.

Note that for every $i \in S$ and $j \in V_1$, $d(i, j) \geq d_{-1}/4$. And $|V_1| \geq \alpha n$. Hence, we have $\Omega(n)$ pairs of voters $(i, j)$ such that $d(i, j) \geq d_{-1}/4$. Further, $d(a_{OPT}, i) + d(a_{OPT}, j) \geq d(i, j)$. Hence, it follows that

$$C(a_{OPT}) = \Omega(n) \cdot d_{-1}. \quad (6)$$

Let $a^* = \hat{f}(I)$. If $a^* = a_{-1} = a_1$, then we have

$$C(a^*) \leq C(a_{OPT}) + n \cdot d(a^*, a_{OPT}) \leq C(a_{OPT}) + n \cdot d_{-1} = O(C(a_{OPT})), \quad (7)$$

yielding a constant upper bound on $\phi(\hat{f}_{\text{fail}}, I) = C(a^*)/C(a_{OPT})$.

On the other hand, if $a^* = a_1$, we have

$$C(a_{OPT}) \geq C(a_1) - n \cdot d(a_1, a_{OPT}) \geq \frac{n \cdot d(a_1, a^*)}{2} - n \cdot d_{-1} \geq \frac{n \cdot d(a_{OPT}, a^*)}{2} - O(C(a_{OPT})). \quad (6)$$

Here, the second transition follows because in the general election, at least $n/2$ voters vote for $a_1$ over $a_1$ and $d(a_1, a_{OPT}) \leq d_{-1}$, and the final transition follows from Equation (6). This implies

$$C(a_{OPT}) = \Omega(n \cdot d(a_{OPT}, a^*)). \quad (7)$$

On the other hand, we have

$$C(a^*) \leq C(a_{OPT}) + n \cdot d(a_{OPT}, a^*) = O(C(a_{OPT})), \quad \text{where the last transition follows due to Equation (7).}$$

Hence, we again have the desired constant upper bound on $\phi(\hat{f}_{\text{fail}}, I)$.

Finally, if $\rho(A) = \{x_1, x_2\}$ (w.l.o.g., $x_1 < x_2$), due to separability, we have two options:

1. $\rho(A_{-1}) = x_1$ and $\rho(A_1) = x_2$: Therefore, $a_{OPT} \in \{a_{-1}, a_1\}$, and since $a^*$ is the majority winner, $a^* = a_{OPT}$, and the distortion is 1.

2. $\rho(A_{-1}) = \{x_1, x_2\}$ and $\rho(A_1) = x_2$: If $a_{OPT} \in \{a_{-1}, a_1\}$, then it is similar to the previous case. Otherwise, this means $\rho(a_{OPT}) = x_1$ and $\rho(a_{-1}) = \rho(a_1) = x_2$. Separability means the voters of party 1 lie in $\{x|x \geq x_2\}$, so $C(a_{OPT}) \geq |V_1| \cdot d(x_1, x_2) \geq \alpha \cdot n \cdot d(x_1, x_2)$. While $C(a^*) \leq C(a_{OPT}) + |V_{-1}| \cdot d(x_1, x_2) \leq C(a_{OPT}) + n(1 - \alpha)d(x_1, x_2)$. Combining these two equations we get the distortion is $\frac{4 - 2\gamma}{\alpha}$.

$\square$

### Allowing Partial Participation in the General Election

While data indicates almost all primary voters vote in the general election (Sides et al. 2018), we consider the possibility that out of all the electorate, only $\gamma n$ vote in the general election.

#### Theorem 10

For any instance of $\mathcal{T} \in \mathcal{T}_{m,M}$, if only $\gamma \cdot n$ ($\gamma \in (0,1]$) voters participate in the general election, creating an instance $I$, $\phi(\hat{f}, I) \leq \frac{4 - 2\gamma}{\gamma} \phi(\hat{f}, \mathcal{T})$

**Proof.** For $\mathcal{T} \in \mathcal{T}_{m,M}$, let the winner of the election be $a^*$, and the optimal candidate $a_{OPT}$, hence $\phi(\hat{f}, \mathcal{T}) = C(a_{OPT})$. For the equivalent $I$ in which only $\gamma \cdot n$ voters participate in the general elections (we shall refer to the electorate in the general election as $V_\gamma$) let the winner of the election be $a^*$, while the optimal candidate does not change, hence $\phi(\hat{f}, I) = C(a^*)$.

Suppose $a^* \neq a_\gamma$, then at least $\frac{n}{2} \gamma$ voters prefer $a^*$, hence $C(a^*) \geq \frac{n\gamma}{2} \cdot d(a^*, a_\gamma) \Rightarrow d(a^*, a_\gamma) \leq \frac{4C(a_\gamma)}{n\gamma}$ (note that this is true even if $a^* = a_\gamma$).

Let us call the set of voters for which $d(i, a^*) < d(i, a_\gamma)$ $X$.

$$C(a^*) = \sum_{i \in X} d(i, a^*) + \sum_{i \notin X} d(i, a^*)$$

$$\leq \sum_{i \in X} d(i, a_\gamma) + \sum_{i \notin X} d(i, a_\gamma) + d(a_\gamma, a^*)$$

$$\leq \sum_{i \in X} d(i, a_\gamma) + \sum_{i \notin X} d(a_\gamma, a^*)$$

$$\leq C(a^*) + n(1 - \frac{\gamma}{2})d(a_\gamma, a^*)$$

Combining our equation regarding $d(a^*, a_\gamma)$ we get

$$C(a^*) \leq C(a_\gamma)(1 + \frac{4 - 2\gamma}{\gamma}) = \frac{4 - \gamma}{\gamma} C(a_\gamma)$$

Hence:

$$\phi(\hat{f}, I) = \frac{C(a^*)}{C(a_{OPT})} \leq \frac{4 - \gamma}{\gamma} \cdot \frac{C(a^*)}{C(a_{OPT})} = \frac{4 - \gamma}{\gamma} \phi(\hat{f}, \mathcal{T})$$

$\square$

---

6 Since voters are the same, $C^T = C^I$, and we are using $C$ to denote the cost.