Discrete Mathematics 312 (2012) 1542–1549

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Balanced group-labeled graphs

Manas Joglekar, Nisarg Shah, Ajit A. Diwan *

Department of Computer Science and Engineering, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India

ARTICLE INFO

Article history:
Available online 13 October 2011

Keywords:
Group-labeled graphs
Signed graphs
Marked graphs
Balanced labellings
Fundamental cycles
Vertex switching
Markable graphs

ABSTRACT

A group-labeled graph is a graph whose vertices and edges have been assigned labels from some abelian group. The weight of a subgraph of a group-labeled graph is the sum of the labels of the vertices and edges in the subgraph. A group-labeled graph is said to be balanced if the weight of every cycle in the graph is zero. We give a characterization of balanced group-labeled graphs that generalizes the known characterizations of balanced signed graphs and consistent marked graphs. We count the number of distinct balanced labellings of a graph by a finite abelian group Γ and show that this number depends only on the order of Γ and not its structure. We show that all balanced labellings of a graph can be obtained from the all-zero labeling using simple operations. Finally, we give a new constructive characterization of consistent marked graphs and markable graphs, that is, graphs that have a consistent marking with at least one negative vertex.

1. Introduction

The study of signed and marked graphs has a long history and diverse applications [11,14]. A signed graph is an undirected graph whose edges are labeled positive or negative. A marked graph is an undirected graph whose vertices are signed. Signed and marked graphs may be considered to be special cases of group-labeled graphs. In this paper, we generalize some results known for signed and marked graphs to group-labeled graphs. This generalization also gives some new properties of signed and marked graphs.

Let Γ be a finite abelian group. A Γ-labeled graph is a graph whose vertices and edges are assigned labels from Γ. A Γ-labeled graph is denoted (G, w), where G is a graph and $w : V(G) \cup E(G) \to \Gamma$ is an assignment of labels from Γ to the vertices and edges of G. Thus a signed graph may be considered to be a Z_2-labeled graph with all vertex labels 0, while a marked graph has all edge labels 0.

One of the central notions in the study of signed graphs is that of balance. A signed graph is said to be balanced if every cycle in the graph contains an even number of edges with negative sign. Similarly, a marked graph is said to be consistent if every cycle contains an even number of vertices with negative sign. We generalize the notions of balance and consistency and their characterizations to group-labeled graphs.

Let (G, w) be a group-labeled graph. The weight of a subgraph H of G, denoted by $w(H)$, is $\sum_{x \in V(H) \cup E(H)} w(x)$. The graph (G, w) is said to be a balanced group-labeled graph and w is a balanced labeling of G if $w(C) = 0$ for all cycles C in G. Thus balanced Z_2-labeled graphs include both balanced signed graphs and consistent marked graphs.

Balanced signed graphs were first characterized by Harary [6] who proved the following theorem.

Theorem 1. *A signed graph G is balanced iff there exists a subset $A \subseteq V(G)$ such that an edge has a negative sign iff it has exactly one end vertex in A.*

* Corresponding author.
E-mail addresses: manas@cse.iitb.ac.in (M. Joglekar), nisarg@cse.iitb.ac.in (N. Shah), aad@cse.iitb.ac.in (A.A. Diwan).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2011.09.021

Characterizations of consistent marked graphs were obtained much later. Initial results were given by Acharya and Rao [1,2,9] and a simpler characterization was obtained by Hoede [7], who proved the following theorem.

Theorem 2. *A connected marked graph G is consistent iff there exists a spanning tree T in G such that*

(i) *every fundamental cycle with respect to T has an even number of vertices with negative sign;*
(ii) *any path in T, which is the intersection of two fundamental cycles, has end vertices with the same sign.*

Subsequently, Roberts and Xu [12] gave several alternative characterizations of consistent marked graphs that were similar to Hoede's characterization.

Our main result in Section 2 is a generalization of Theorem 2 to group-labeled graphs. The characterizations of Roberts and Xu also extend to group-labeled graphs in a natural way. In Section 3, we give a simple formula for the number of balanced labellings of a graph by a finite abelian group Γ. An interesting observation is that this number depends only on the order of Γ and not its structure.

Another characterization of balanced signed graphs is in terms of the notion of switching [13]. A switch applied to a vertex of a signed graph changes the sign of edges incident with it, keeping the signs of all other edges the same. A switch applied to any vertex of a balanced signed graph gives a balanced signed graph. It follows easily from Theorem 1 that a signed graph is balanced iff all edges can be made positive by switching a subset of vertices.

In Section 4, we define some simple operations on group-labeled graphs that preserve balance, and show that a group-labeled graph is balanced iff all edge and vertex labels can be made 0 using these operations. These operations generalize the switching operation for signed graphs.

Beineke and Harary [4] characterized directed graphs whose vertices can be consistently marked such that at least one vertex has a '−' sign, and in [5], they left open the same question for undirected graphs. Graphs that admit such a marking are called *markable*. Roberts [10] characterized 2-connected markable graphs in which the longest cycle has length at most five. In Section 5, we give a new constructive characterization of consistent marked graphs, which immediately gives a simple constructive characterization of markable graphs.

The notation and terminology used is largely standard, however, we clarify a few terms that are frequently required. The graphs that we consider are finite and undirected but may have loops and/or multiple edges. We assume that each edge has two end vertices that may possibly be the same, in which case it is a loop, else it is a link. The vertex set of a graph G is denoted by $V(G)$ and the edge set by $E(G)$.

A path P is a sequence of distinct vertices and edges $v_0, e_1, v_1, \ldots, e_l, v_l$ such that the end vertices of e_i are v_{i-1} and v_i for $1 \leq i \leq l$. The path P is said to be a v_0–v_l path of length l. The end vertices of P are v_0 and v_l, all other vertices are internal. If $A \subset V(G)$ and $v \in V(G) \setminus A$, then an A–v path is a u–v path for some $u \in A$, whose internal vertices are not in A. A cycle C is a sequence of vertices and distinct edges $v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l$, such that $v_0 = v_l$ and all other vertices are distinct, and e_i has end vertices v_{i-1} and v_i, for $1 \leq i \leq l$.

A tree T in a graph G is a connected subgraph of G without cycles. If u, v are vertices in a tree T, we denote by $T[u, v]$ the unique path contained in T having end vertices u and v. In particular, if P is a path and u, v are vertices in P, then $P[u, v]$ is the subpath of P with end vertices u, v. If T is a spanning tree in a graph G, a fundamental cycle with respect to T is a cycle in G containing exactly one edge not in T. The edges not in T are called cotree edges. If $A \subset V(G)$ is a non-empty proper subset of vertices, then the set of edges having exactly one end vertex in A is called a cut in the graph.

If G_1, G_2 are graphs then the graph $G_1 \cup G_2$ has vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 is a subgraph of a graph G and uv is an edge in G, let $G_1 \cup uv$ be the graph with vertex set $V(G_1) \cup \{u\} \cup \{v\}$, and edge set $E(G_1) \cup \{uv\}$.

In this paper, we will consider group-labeled graphs (G, w) whose vertices and edges are assigned labels from some arbitrary finite abelian group Γ. However, if $\Gamma \cong \Gamma_1 \times \Gamma_2$ for some non-trivial groups Γ_1 and Γ_2, then a labeling of G by Γ may be considered to be a product of labelings by Γ_1 and Γ_2. The labeling by Γ is balanced iff the corresponding labelings by Γ_1 and Γ_2 are balanced, and the number of balanced labelings by Γ is the product of the number of balanced labelings by Γ_1 and Γ_2. Therefore, to understand balanced group-labeled graphs, it is sufficient to consider the case when Γ is a cyclic group, in particular Z_k, for some positive integer k. Further, there is no loss of generality in assuming the graphs being considered are connected. Thus, in the rest of the paper, a group-labeled graph (G, w) will be a connected graph G whose vertices and edges have been assigned labels from Z_k, for some fixed positive integer k.

We will assume elements of Z_k to be $\{0, 1, 2, \ldots, k-1\}$. However, sometimes we need to consider the labels to be integers rather than elements of Z_k. In order to do this conveniently, we define the functions $\pi : Z \to Z_k$ and $\pi' : Z_k \to Z$, where $\pi(i) = i \bmod k$ for $i \in Z$ and $\pi'(j) = j$ for $j \in Z_k$. Also, we will use $\oplus$ ($\ominus$) to denote addition (subtraction) modulo k and $+$ ($-$) to denote ordinary integer addition (subtraction).

2. Characterization

In this section, we generalize the characterizations of balanced signed graphs (Theorem 1) and consistent marked graphs (Theorem 2) to group-labeled graphs.

Definition 1. Let (G, w) be a group-labeled graph. A u–v path P in G is said to be *good* if $2w(P) = w(u) \oplus w(v)$. An edge uv is said to be *good* if the path of length 1 containing the edge uv is good, that is if $2w(uv) \oplus w(u) \oplus w(v) = 0$. An edge or path that is not good is said to be *bad*.

Lemma 1. *Let (G, w) be a group-labeled graph. Let P be a u–v path in G and let x be any vertex in P. Suppose $P[u, x]$ is a good path. Then P is good iff $P[x, v]$ is good.*

Proof. Since $P[u, x]$ is good, $2w(P[u, x]) = w(u) \oplus w(x)$. Then $2w(P) = 2(w(P[u, x]) \oplus w(P[x, v]) \ominus w(x)) = 2w(P[x, v]) \oplus w(u) \ominus w(x)$. Therefore, $2w(P) = w(u) \oplus w(v)$ iff $2w(P[x, v]) = w(x) \oplus w(v)$. □

Lemma 2. *Let (G, w) be a group-labeled graph and let P be any u–v path in G. Let x_1, y_1 be two distinct vertices in P such that $|P[u, x_1]| < |P[u, y_1]|$. Let Q be an x_1–y_1 path in G that is internally vertex-disjoint from P. Suppose Q and $P[x_1, y_1]$ are good paths. Then P is good iff the path $P' = P[u, x_1] \cup Q \cup P[y_1, v]$ is good.*

Proof. This follows from the fact that $2w(P) \ominus 2w(P') = 2w(P[x_1, y_1]) \ominus 2w(Q) = 0$, since $2w(P[x_1, y_1]) = 2w(Q) = w(x_1) \oplus w(y_1)$. □

Lemma 3. *Let (G, w) be a group-labeled graph in which all edges are good, that is, $2w(uv) \oplus w(u) \oplus w(v) = 0$ for all edges uv. Then (G, w) is balanced iff one of the following holds.*

(i) *k is odd and for every edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2)$, where $\delta(u, v) = k$ if $\pi'(w(u)) + \pi'(w(v))$ is odd and $\delta(u, v) = 0$ if $\pi'(w(u)) + \pi'(w(v))$ is even.*
(ii) *k is even and there exists a subset $A \subseteq V(G)$ such that for every edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u) + \delta(v))/2)$, where $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$ if $x \notin A$, for all $x \in V(G)$.*

Proof. First suppose (G, w) is balanced. If k is odd and every edge uv is good, for any given values of $w(u)$ and $w(v)$, there is a unique value of $w(u, v)$ that satisfies $2w(u, v) \oplus w(u) \oplus w(v) = 0$. This value is exactly $\pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2)$.

Suppose k is even. Since all edges are good, $w(u)$ and $w(v)$ must have the same parity for all edges uv. Since G is connected, all vertex labels must have the same parity. Given $w(u)$ and $w(v)$ having the same parity, there are two possible values of $w(uv)$ that satisfy $2w(uv) \oplus w(u) \oplus w(v) = 0$; either $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)))/2)$ or $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + k)/2)$. Since every edge in G is good, by Lemma 1, every path in G is good. A good u–v path P is said to be *small* if $w(P) = \pi((\pi'(w(u)) + \pi'(w(v)))/2)$ and *large* if $w(P) = \pi((\pi'(w(u)) + \pi'(w(v)) + k)/2)$. Let T be any spanning tree in G and let r be any fixed vertex in G. Let $A \subseteq V(G)$ be the set of vertices v such that $T[r, v]$ is small. We show that for any edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u) + \delta(v))/2)$, where $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$ if $x \notin A$. Note that $r \in A$ and for any vertex v, $w(T[r, v]) = \pi((\pi'(w(r)) + \pi'(w(v)) + \delta(v))/2)$, by definition.

Let uv be any edge in T. We may assume without loss of generality, that $T[r, v]$ contains u and hence $w(T[r, v]) = w(T[r, u]) \oplus w(uv) \oplus w(v)$. Then $w(uv) = w(T[r, v]) \ominus w(T[r, u]) \ominus w(v) = \pi((\pi'(w(r)) + \pi'(w(v)) + \delta(v))/2) \ominus \pi((\pi'(w(r)) + \pi'(w(u)) + \delta(u))/2) \ominus w(v) = \pi((\pi'(w(v)) - \pi'(w(u)) + \delta(v) - \delta(u))/2 - \pi'(w(v))) = \pi(-(\pi'(w(v)) + \pi'(w(u)) + \delta(v) + \delta(u))/2)$.

Let uv be an edge not in T. Since w is a balanced labeling, $w(uv) \oplus w(T[u, v]) = 0$. Let $T[r, x] = T[r, u] \cap T[r, v]$. Then $w(T[u, v]) = w(T[r, u]) \oplus w(T[r, v]) \ominus 2w(T[r, x]) \oplus w(x) = \pi((\pi'(w(r)) + \pi'(w(u)) + \delta(u))/2) \oplus \pi((\pi'(w(r)) + \pi'(w(v)) + \delta(v))/2) \ominus w(r) \ominus w(x) \oplus w(x) = \pi((\pi'(w(u)) + \pi'(w(v)) + \delta(u) + \delta(v))/2)$.

Conversely, suppose (G, w) is a group-labeled graph in which all edges are good. Suppose k is odd, and for every edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2)$. In any cycle C, there are an even number of edges uv such that $\delta(u, v) = k$. Then the weight of any cycle $C = v_0, e_1, \ldots, e_l, v_l$, is

$$\begin{aligned}\pi\left(\sum_{1\le i\le l} \pi'(w(v_i)) + \pi'(w(e_i))\right) &= \pi\left(\sum_{1\le i\le l} \pi'(w(v_i)) - (\pi'(w(v_{i-1})) + \pi'(w(v_i)) + \delta(v_{i-1}, v_i))/2\right)\\ &= \pi\left(\sum_{1\le i\le l} -\delta(v_{i-1}, v_i)/2\right)\\ &= 0.\end{aligned}$$

Thus (G, w) is balanced.

Suppose k is even and there exists such a subset A. Then the weight of the cycle C is

$$\begin{aligned}\pi\left(\sum_{1\le i\le l} \pi'(w(v_i)) + \pi'(w(e_i))\right) &= \pi\left(\sum_{1\le i\le l} \pi'(w(v_i)) - (\pi'(w(v_{i-1})) + \pi'(w(v_i)) + \delta(v_{i-1}) + \delta(v_i))/2\right)\\ &= \pi\left(\sum_{1\le i\le l} -\delta(v_i)\right)\\ &= 0.\end{aligned}$$

Thus (G, w) is balanced. □

Note that Lemma 3 can be considered to be a generalization of the characterization of balanced signed graphs, since if all vertex labels in a Z_2-labeling are 0, then all edges are good.

Definition 2. A spanning tree T in a group-labeled graph (G, w) is *good* if

(i) every fundamental cycle with respect to T has weight 0;
(ii) any path in T, which is the intersection of two distinct fundamental cycles with respect to T, is good.

Lemma 4. *Let (G, w) be a group-labeled graph and suppose it contains a good spanning tree T. Let P_1, P_2, P_3 be 3 edge-disjoint paths between vertices u and v in G. Then P_1, P_2, P_3 are good paths in G.*

Proof. Let H be the graph $P_1 \cup P_2 \cup P_3$. The proof is by induction on the number of cotree edges in H, that is edges in H that are not in T.

Case 1. First suppose that the paths are not internally vertex-disjoint. Without loss of generality, let $x \notin \{u, v\}$ be a vertex in $V(P_1) \cap V(P_2)$. At least one of the paths $\{P_1[x, v], P_2[x, v]\}$, say $P_1[x, v]$, contains a cotree edge. Let P' be a u–x path contained in $P_3 \cup P_2[x, v]$. Then $P_1[u, x], P_2[u, x], P'$ are 3 edge-disjoint u–x paths in G such that the number of cotree edges in the union of the three paths is less than the number in H. By induction, $P_1[u, x], P_2[u, x], P'$ are good paths. By a symmetrical argument, $P_1[x, v], P_2[x, v], P''$ are good paths, where P'' is an x–v path contained in $P_3 \cup P_1[u, x]$ or $P_3 \cup P_2[u, x]$. By Lemma 1, P_1 and P_2 are good paths. If P_3 has any internal vertex in common with either P_1 or P_2, the same argument implies that P_3 is a good path. If P_3 is internally vertex-disjoint from P_1 and P_2, since P' and $P_2[x, v]$ are good paths, by Lemma 1, P_3 is also a good path.

Case 2. Suppose the paths are internally vertex-disjoint.

Case 2.1. Suppose there is a cotree edge xy in H such that the path $T[x, y]$ is not contained in H. Then we can find two distinct vertices x_1, y_1 in $T[x, y]$ such that $V(T[x_1, y_1]) \cap V(H) = \{x_1, y_1\}$ and $E(T[x_1, y_1]) \cap E(H) = \emptyset$.

Case 2.1.1. Suppose $\{x_1, y_1\} \subseteq V(P_1)$. We may assume without loss of generality that $|P_1[u, x_1]| < |P_1[u, y_1]|$. Let P_1' be the path $P_1[u, x_1] \cup T[x_1, y_1] \cup P_1[y_1, v]$. Then the number of cotree edges in $P_1' \cup P_2 \cup P_3$ is less than the number in H and by induction P_1', P_2, P_3 are good paths. Since at least one of $\{P_2, P_3\}$, say P_3, contains a cotree edge, by induction, $P_1[x_1, y_1]$, $T[x_1, y_1]$, $P_1[u, x_1] \cup P_2 \cup P_1[y_1, v]$ are also good paths. By Lemma 2, P_1 is also a good path. A symmetrical argument holds if $\{x_1, y_1\} \subseteq V(P_i)$ for $i \in \{2, 3\}$.

Case 2.1.2. Suppose $x_1 \in V(P_1)$ and $y_1 \in V(P_2)$. Suppose both $P_1[u, x_1]$ and $P_1[x_1, v]$ contain cotree edges. Then, by induction, the 3 edge-disjoint y_1–v paths $P_2[y_1, v]$, $P_2[u, y_1] \cup P_3$, $T[x_1, y_1] \cup P_1[x_1, v]$ are good paths. By a symmetrical argument, $P_2[u, y_1]$, $P_2[y_1, v] \cup P_3$, $T[x_1, y_1] \cup P_1[u, x_1]$ are good u–y_1 paths. Lemma 1 now implies that P_2 and P_3 are also good paths. If P_3 contains a cotree edge, by induction, the 3 edge-disjoint x_1–y_1 paths $T[x_1, y_1]$, $P_1[u, x_1] \cup P_2[u, y_1]$, $P_1[x_1, v] \cup P_2[y_1, v]$ are good paths. By Lemma 1, $P_1[u, x_1]$ is good and so is $P_1[x_1, v]$. Applying Lemma 1 again, we can conclude that P_1 is a good path. If P_3 does not contain a cotree edge, either $P_2[u, y_1]$ or $P_2[y_1, v]$ must contain a cotree edge. We may assume, by symmetry, that $P_2[y_1, v]$ contains a cotree edge. Then, by induction, the 3 edge-disjoint u–x_1 paths $P_1[u, x_1]$, $P_2[u, y_1] \cup T[x_1, y_1]$, $P_3 \cup P_1[x_1, v]$ are good. By repeated application of Lemma 1, we conclude that $T[x_1, y_1]$ and P_1 are good paths.

A symmetrical argument holds if both $P_2[u, y_1]$ and $P_2[y_1, v]$ contain a cotree edge. The only other possibility, taking into account symmetry, is that $P_1[u, x_1]$, $P_2[y_1, v]$ and P_3 contain a cotree edge. Then, by induction, we can conclude that the paths $P_2[y_1, v]$, $T[x_1, y_1] \cup P_1[x_1, v]$, $P_2[u, y_1] \cup P_3$, $P_1[u, x_1]$, $P_2[u, y_1] \cup T[x_1, y_1]$, $P_1[x_1, v] \cup P_3$, $T[x_1, y_1]$, $P_1[u, x_1] \cup P_2[u, y_1]$, $P_1[x_1, v] \cup P_2[y_1, v]$ are good paths. Again by repeated application of Lemma 1, we conclude that P_1, P_2, P_3 are good paths. A symmetrical argument holds if $x_1 \in V(P_i)$ and $y_1 \in V(P_j)$ for some $i \neq j$, $\{i, j\} \subset \{1, 2, 3\}$.

Case 2.2. Finally, suppose that for every cotree edge xy in H, $T[x, y]$ is contained in H. This is possible if and only if one of the paths P_1, P_2, P_3 is contained in T and each of the other two paths contains exactly one cotree edge. We may assume without loss of generality that P_1, P_2 contain exactly one cotree edge and P_3 is contained in T. Then $C_1 = P_1 \cup P_3$ and $C_2 = P_2 \cup P_3$ are fundamental cycles with respect to T, whose intersection is the path P_3. Since T is good, P_3 is good and $w(C_1) = w(C_2) = 0$. Since $w(C_1) = w(P_1) \oplus w(P_3) \ominus w(u) \ominus w(v) = 0$ and $2w(P_3) = w(u) \oplus w(v)$, we get $2w(P_1) = 2(w(u) \oplus w(v)) \ominus 2w(P_3) = w(u) \oplus w(v)$. Thus P_1 is good and by the same argument P_2 is also good.

This completes the proof of Lemma 4. $\square$

Theorem 3. *A connected group-labeled graph (G, w) is balanced iff it contains a good spanning tree. If (G, w) contains a good spanning tree, then all spanning trees in (G, w) are good.*

Proof. Suppose (G, w) is a balanced group-labeled graph and let T be any spanning tree in G. By definition, every fundamental cycle with respect to T has weight 0. Let P be a path in T that is the intersection of two fundamental cycles C_1 and C_2. Let u, v be the end vertices of P. The symmetric difference of C_1 and C_2 is a cycle C_3 in G. Then $w(C_3) = w(C_1) \oplus w(C_2) \ominus 2w(P) \oplus w(u) \oplus w(v)$. Since $w(C_i) = 0$ for $1 \leq i \leq 3$, we have $2w(P) = w(u) \oplus w(v)$ and P is a good path. Thus every spanning tree in G is good.

Conversely, let T be any good spanning tree in (G, w). We show that every cycle C in G has weight 0. The proof is by induction on the number of cotree edges in C.

If there is only one cotree edge in C, then C is a fundamental cycle with respect to T and has weight 0, since T is good.

If there are at least two cotree edges in C, then there exist two distinct vertices $x, y \in V(C)$ such that $V(T[x, y]) \cap V(C) = \{x, y\}$ and $E(T[x, y]) \cap E(C) = \emptyset$. Let P_1, P_2 be the two edge-disjoint x–y paths in C. Then, by Lemma 4, $P_1, P_2, T[x, y]$ are good paths. Also, each of P_1, P_2 contains at least one cotree edge. Then, by induction, the cycles $C_1 = P_1 \cup T[x, y]$ and $C_2 = P_2 \cup T[x, y]$ have weight 0. Hence $w(C) = w(P_1) \oplus w(P_2) \ominus w(x) \ominus w(y) = w(C_1) \ominus w(T[x, y]) \oplus w(x) \oplus w(y) \oplus w(C_2) \ominus w(T[x, y]) \oplus w(x) \oplus w(y) \ominus w(x) \ominus w(y) = w(C_1) \oplus w(C_2) \ominus 2w(T[x, y]) \oplus w(x) \oplus w(y) = 0$. Thus C also has weight 0. Hence (G, w) is balanced and every spanning tree in G is good. □

Note that from Lemma 4 and Theorem 3, we can conclude the following corollary.

Corollary 1. *If (G, w) is a balanced group-labeled graph and P_1, P_2, P_3 are 3 edge-disjoint u–v paths in G, then P_1, P_2, P_3 are good paths.*

The characterizations given by Roberts and Xu [12] can also be generalized in a similar way. Since the proofs are almost the same as in the case of marked graphs, we mention only one, which follows immediately from Theorem 3.

Corollary 2. *A connected group-labeled graph (G, w) is balanced iff there is a spanning tree T in G such that all cycles containing at most two cotree edges have weight 0.*

3. Counting

In this section, we give a simple formula for the number of distinct balanced labellings of a graph G by Z_k, and hence the number of balanced labellings of G by an arbitrary finite abelian group Γ.

Definition 3. A 3-edge-connected component of a graph G is a maximal subset of vertices of G, such that for any two vertices in the subset there are 3 edge-disjoint paths between them in G.

Note that the existence of 3 edge-disjoint paths between a pair of vertices is an equivalence relation on the set of vertices, and the 3-edge-connected components of the graph are the equivalence classes of this relation. Thus distinct 3-edge-connected components of a graph are disjoint and partition the vertex set. Let $c_3(G)$ denote the number of 3-edge-connected components of a graph G. Let $W(G, k)$ denote the set of balanced labellings of G by Z_k and let $w(G, k) = |W(G, k)|$.

Theorem 4. *For any connected graph G and positive integer k*

$$w(G, k) = k^{|G|+c_3(G)-1}.$$

Proof. The proof is by induction on $|G|$.

Case 1. Suppose there is cut X in G of size 1. Let G_1 and G_2 be the components of $G - X$. Then a balanced labeling of G is obtained from balanced labellings of G_1 and G_2 by assigning an arbitrary label to the edge in X. Thus $w(G, k) = w(G_1, k) \times w(G_2, k) \times k$. Since $|G| = |G_1| + |G_2|$ and $c_3(G) = c_3(G_1) + c_3(G_2)$, the theorem follows by induction.

Case 2. Suppose G is 2-edge-connected and has a 2-edge-cut X. Let G_1 and G_2 be the components of $G - X$. Let $X = \{p_1p_2, q_1q_2\}$ such that $p_i, q_i \in V(G_i)$ for $i \in \{1, 2\}$. Note that p_i may be the same vertex as q_i. Let G_i' be the graph obtained from G_i by adding a new edge e_i with end vertices $\{p_i, q_i\}$ for $i \in \{1, 2\}$. Then $|G| = |G_1'| + |G_2'|$ and $c_3(G) = c_3(G_1') + c_3(G_2')$. It is sufficient to show that $w(G, k) = w(G_1', k) \times w(G_2', k) \times k$. The theorem then follows by induction.

We show a bijection $F : W(G, k) \to W(G_1', k) \times W(G_2', k) \times Z_k$. Let w be any balanced labeling of G by Z_k. Let w_i be the labeling of G_i' defined by $w_i(x) = w(x)$ for $x \in V(G_i) \cup E(G_i)$ and $w_i(e_i) = w(P_{3-i}) \oplus w(p_1p_2) \oplus w(q_1q_2)$, where P_j is any p_j–q_j path in G_j. Note that since w is balanced, all p_j–q_j paths in G_j must have the same weight, so w_i is well-defined. This also implies that w_i is a balanced labeling of G_i'. Define $F(w) = (w_1, w_2, w(p_1p_2))$.

Conversely, suppose w_1 and w_2 are balanced labellings of G_1' and G_2', respectively, and let $a \in Z_k$. We show that there is a unique balanced labeling w of G such that $F(w) = (w_1, w_2, a)$. Define $w(x) = w_i(x)$ for all $x \in V(G_i) \cup E(G_i)$ and $w(p_1p_2) = a$. Let $w(q_1q_2) = w_1(e_1) \oplus w_2(e_2) \ominus a$. It is then easy to check that w is a balanced labeling of G and $F(w) = (w_1, w_2, a)$.

Case 3. Suppose G is 3-edge-connected and let w be any balanced labeling of G by Z_k. For every edge uv in G, there are 2 edge-disjoint u–v paths in $G - uv$, hence Corollary 1 implies all edges are good, and $2w(uv) \oplus w(u) \oplus w(v) = 0$.

By Lemma 3, if k is odd, then for any edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2)$, where $\delta(u, v) = 0$ if $\pi'(w(u)) + \pi'(w(v))$ is even and $\delta(u, v) = k$, otherwise. Since the labels of the vertices can be arbitrary, the number of distinct balanced labellings is $k^{|G|}$. Since $c_3(G) = 1$, the theorem holds.

If k is even, all vertex labels must have the same parity. Again by Lemma 3, there exists a subset A of vertices such that for any edge uv, $w(uv) = \pi(-(\pi'(w(u)) + \pi'(w(v)) + \delta(u) + \delta(v))/2)$, where $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$, otherwise. The labeling obtained is the same if the set A is replaced by $V(G) \setminus A$. Thus the total number of balanced labellings is $2 \times (k/2)^{|G|} \times 2^{|G|-1} = k^{|G|}$.

This completes the proof of Theorem 4. □

The following corollary of Theorem 4 follows immediately by induction.

Corollary 3. *The number of distinct balanced labellings of a connected graph G by an arbitrary finite abelian group Γ is* $|\Gamma|^{|G|+c_3(G)-1}$.

It may be noted that in [3], edge weightings by real numbers in which all cycles have total weight 0 were considered. It was shown that they form a vector space of dimension $c_3(G) - 1$. If vertex labels are also allowed, the dimension becomes $|G| + c_3(G) - 1$. This can be proved in the same way as Theorem 4.

4. Switching

In this section, we define a few simple operations on group-labeled graphs, called shifting, that preserve balance, and show that any balanced labeling of a graph can be obtained from the all-zero labeling using these operations. Let $(G, \mathbf{0})$ denote the group-labeled graph G with all vertex and edge labels 0.

Definition 4. We define the following operations on a group-labeled graph (G, w). Here a denotes an arbitrary element of Z_k.

(i) Let u be any vertex in G. Let (G, w') be the group-labeled graph defined by $w'(u) = w(u) \oplus 2a$, $w'(e) = w(e) \ominus 2a$ if e is a loop incident with u, $w'(e) = w(e) \ominus a$ if e is a link incident with u, and $w'(x) = w(x)$ for all other vertices and edges x in G. (G, w') is said to be obtained from (G, w) by shifting the vertex u by a.
(ii) Let X be a minimal cut of size at most two in G. Let e_1 be an edge in X and let G_1 be a component of $G - X$. Let (G, w') be the group-labeled graph defined by $w'(v) = w(v) \oplus a$ for all vertices $v \in V(G_1)$, $w'(e) = w(e) \ominus a$ for all edges $e \in E(G_1) \cup \{e_1\}$ and $w'(x) = w(x)$ for all other vertices and edges x in G. (G, w') is said to be obtained from (G, w) by shifting the edge e_1 in the cut X by a.
(iii) Let (G, w') be the group-labeled graph defined by $w'(v) = w(v) \oplus a$ for all vertices $v \in V(G)$ and $w'(e) = w(e) \ominus a$ for all edges $e \in E(G)$. (G, w') is said to be obtained from (G, w) by shifting the graph G by a.

Note that if $k = 2$, then switching a vertex is equivalent to shifting it by 1. A group-labeled graph (G, w) is said to be shift equivalent to the graph (G, w') if (G, w') can be obtained from (G, w) by a sequence of shift operations.

Theorem 5. *A group-labeled graph (G, w) is balanced iff it is shift equivalent to $(G, \mathbf{0})$.*

Proof. It is easy to verify that each of the shifting operations preserves balance. Thus any group-labeled graph (G, w) that is shift equivalent to $(G, \mathbf{0})$ is balanced. To prove the converse, suppose (G, w) is a balanced group-labeled graph.

Let (G, w') be a group-labeled graph shift equivalent to (G, w) such that the order of the largest connected component of the spanning subgraph of (G, w') consisting of good edges is as large as possible and let H be this component.

Case 1. Suppose H spans G. We claim that all edges in (G, w') are good. If uv is a bad edge, let P be a u–v path in H consisting of good edges. Since the cycle $P \cup uv$ has weight 0, $2w(uv) \oplus 2w(P) = 0$ and since P is a good path, $2w(P) = w(u) \oplus w(v)$. This implies uv is a good edge, a contradiction.

If all edges are good, and k is odd, then by Lemma 3, for every edge uv, $w'(uv) = \pi(-(\pi'(w'(u))+\pi'(w'(v))+\delta(u, v))/2)$. Thus $(G, \mathbf{0})$ can be obtained from (G, w') by shifting each vertex x by $\pi(-(\pi'(w'(x)))/2)$ if $\pi'(w'(x))$ is even, and shifting it by $\pi(-(\pi'(w'(x)) + k)/2)$ if $\pi'(w'(x))$ is odd. If k is even and all edges are good, all vertices have labels with the same parity, and by Lemma 3, there is a subset $A \subseteq V(G)$ such that for every edge uv, $w'(uv) = \pi(-(\pi'(w'(u)) + \pi'(w'(v)) + \delta(u) + \delta(v))/2)$, where for $x \in V(G)$, $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$ if $x \notin A$. If $\pi'(w'(x))$ is even for all vertices x, then $(G, \mathbf{0})$ can be obtained from (G, w') by shifting vertex x by $\pi(-(\pi'(w'(x)) + \delta(x))/2)$ for all vertices x. If $\pi'(w'(x))$ is odd for all vertices x, then $(G, \mathbf{0})$ can be obtained from (G, w') by first shifting the graph G by 1 and then shifting vertex x by $\pi(-(\pi'(w'(x)) + 1 + \delta(x))/2)$ for all vertices x. This implies (G, w) is shift equivalent to $(G, \mathbf{0})$.

Case 2. Suppose $V(H) \subset V(G)$. Since G is connected, there exists an edge uv in G such that $u \in V(H)$ and $v \notin V(H)$. Clearly, uv is a bad edge. We claim that $G - uv$ does not contain 2 edge-disjoint $V(H)$–v paths. Suppose there exist two such paths P_1, P_2 whose end vertices in $V(H)$ are u_1 and u_2 (not necessarily distinct). Then there is a vertex $x \in V(H)$ such that H contains 3 edge-disjoint paths Q_1, Q_2, Q_3 such that Q_i is an x–u_i path for $i \in \{1, 2\}$ and Q_3 is an x–u path. The vertex x may possibly be one of $\{u_1, u_2, u\}$. Then $Q_3 \cup uv$, $Q_1 \cup P_1$, and $Q_2 \cup P_2$ are 3 edge-disjoint x–v paths in G. By Corollary 1, $Q_3 \cup uv$ is a good path, and since Q_3 is also good, by Lemma 1, the edge uv is also good, a contradiction.

Therefore, there exists a minimal cut X of size at most two that separates $V(H)$ and v and H is a subgraph of some component of $G - X$. Let (G, w'') be the group-labeled graph obtained from (G, w') by shifting the edge uv contained in the cut X by $2w'(uv) \oplus w'(u) \oplus w'(v)$. Then all edges in H as well as the edge uv are good in (G, w''). But this contradicts the choice of the graph (G, w') and the component H.

This completes the proof of Theorem 5. □

5. Markable graphs

In this section, we give a new constructive characterization of consistent marked graphs. This immediately gives a characterization of markable graphs, that is, graphs for which there exists a consistent marking of the vertices with at least one vertex having a '−' sign.

Theorem 6. *A 2-edge-connected marked graph G is consistent iff it satisfies one of the following properties.*

(i) *G is 3-edge-connected with all vertices positive.*
(ii) *G is bipartite with all vertices negative.*
(iii) *G is obtained from a 3-edge-connected graph G' by subdividing exactly once all edges in some cut of G'. The vertices in $V(G) \cap V(G')$ are positive and all other vertices in G, which have degree 2, are negative.*
(iv) *G is obtained from the disjoint union of consistently marked 2-edge-connected graphs G_1 and G_2, by replacing edges $p_1q_1 \in E(G_1)$ and $p_2q_2 \in E(G_2)$ by edges p_1p_2 and q_1q_2.*
(v) *G is obtained from the disjoint union of consistently marked 2-edge-connected graphs G_1 and G_2, by deleting a negative vertex p_i of degree 2 in G_i and adding edges q_1q_2 and r_1r_2, where q_i, r_i are the neighbors of p_i in G_i, for $i \in \{1, 2\}$.*

Proof. It is easy to verify that a marked graph satisfying any one of the conditions in Theorem 6 is consistent.

Now we prove the converse. Suppose G is a 2-edge-connected graph and let w be any consistent marking of G.

Suppose there is a 2-edge-cut $X = \{p_1p_2, q_1q_2\}$ in G. Let G_1, G_2 be the components of $G - X$ such that $\{p_i, q_i\} \subseteq V(G_i)$. Since w is a consistent marking of G, the parity of the number of negative vertices on all p_1–q_1 paths in G_1 and p_2–q_2 paths in G_2 must be the same.

Suppose all p_i–q_i paths in G_i have even number of negative vertices. Let G_i' be the graph obtained from G_i be adding an edge e_i with end vertices p_i and q_i. Then G_i' is 2-edge-connected, and the marking w_i of G_i' obtained by restricting w to $V(G_i)$ is a consistent marking of G_i'. Thus G satisfies property (iv) in Theorem 6.

Suppose all p_i–q_i paths in G_i have odd number of negative vertices. Suppose both G_1 and G_2 contain at least two vertices. Let G_1' and G_2' be the graphs obtained from G by contracting the subgraphs G_2 and G_1, and let v_2 and v_1 be the vertices representing the subgraphs G_2 and G_1 in G_1' and G_2', respectively. Then the marking w_1 of G_1' defined as $w_1(v) = w(v)$ for all $v \in V(G_1)$ and $w_1(v_2) =$ '$-$' is a consistent marking of G_1'. Similarly, $w_2(v) = w(v)$ for all $v \in V(G_2)$ and $w_2(v_1) =$ '$-$' is a consistent marking of G_2'. Since v_1, v_2 have degree 2 in G_2' and G_1', G satisfies property (v) in Theorem 6.

We may now assume that if there is a 2-edge-cut X in G, at least one component of $G - X$ is a single negative vertex. This also implies that G does not contain a positive vertex of degree 2.

Suppose two negative vertices of degree 2 are adjacent in G. If $|G| > 3$ then we get a 2-edge-cut X in G such that each component of $G - X$ contains at least two vertices, a contradiction. If $|G| \leq 3$ then either G contains 2 negative vertices joined by two edges or G is K_3 with two negative and one positive vertex. In the first case, G satisfies property (ii) in Theorem 6, while in the second, there is a positive vertex of degree 2, a contradiction.

So we may assume that no two negative vertices of degree 2 are adjacent. Let G' be the graph obtained from G by replacing every negative vertex v of degree 2 by an edge having the neighbors of v as end vertices (the two neighbors of v need not be distinct). Then G' is 3-edge-connected.

Let (G', w') be the Z_2-labeled graph, defined by $w'(v) = 0$ if $w(v) =$ '$+$' and $w'(v) = 1$ if $w(v) =$ '$-$' for all $v \in V(G')$, and $w'(e) = 0$ for edges $e \in E(G)$ and $w'(e) = 1$ for the added edges, that is for $e \in E(G') \setminus E(G)$. Then (G', w') is a balanced 3-edge-connected Z_2-labeled graph. By Corollary 1, all edges are good and hence all vertex labels have the same parity.

Suppose all vertex labels are 0. Then by Lemma 3, there exists a subset $A \subseteq V(G')$ such that $w'(uv) = 1$ iff exactly one of u, v is in A. If $A = \emptyset$ or $A = V(G')$, then there is no edge labeled 1, and G is a 3-edge-connected graph with all vertices positive. Thus G satisfies property (i) in Theorem 6. If $\emptyset \subset A \subset V(G')$, then G satisfies property (iii) in Theorem 6.

If all vertex labels are 1, then there exists a subset $A \subseteq V(G')$ such that $w'(uv) = 0$ iff exactly one of u, v is in A. In this case, G is a bipartite graph with all vertices negative, and it satisfies property (ii) in Theorem 6. $\square$

Theorem 6 leads to a linear-time algorithm to test whether a marked graph is consistent. This follows from the linear-time algorithm for finding the triconnected components of a graph [8], with a few simple modifications. As a corollary of Theorem 6, we get a constructive characterization of all markable 2-edge-connected graphs.

Corollary 4. *A 2-edge-connected graph G is markable iff it satisfies one of the following properties.*

(i) *G is bipartite.*
(ii) *G is obtained from a 3-edge-connected graph G' by subdividing exactly once all edges in some cut of G'.*
(iii) *G is obtained from the disjoint union of a markable 2-edge-connected graph G_1 and an arbitrary 2-edge-connected graph G_2, by replacing edges $p_1q_1 \in E(G_1)$ and $p_2q_2 \in E(G_2)$, by edges p_1p_2 and q_1q_2.*
(iv) *G is obtained from the disjoint union of two markable 2-edge-connected graphs G_1 and G_2, by deleting a vertex p_i in G_i of degree 2, and adding edges q_1q_2 and r_1r_2, where q_i, r_i are the neighbors of p_i in G_i, for $i \in \{1, 2\}$.*

Acknowledgments

We thank the referees and Thomas Zaslavsky for their helpful comments. We also thank the organizers of the International Conference on Recent Trends in Graph Theory and Combinatorics, Cochin (2010), for the opportunity to present this work at the conference.

References

[1] B.D. Acharya, A characterization of consistent marked graphs, Nat. Acad. Sci. Lett. 6 (1983) 431–440.
[2] B.D. Acharya, Some further properties of consistent marked graphs, Indian J. Pure Appl. Math. 15 (1984) 837–842.
[3] R. Balakrishnan, N. Sudharsanam, Cycle-vanishing edge valuations of a graph, Indian J. Pure Appl. Math. 13 (3) (1982) 313–316.
[4] L.W. Beineke, F. Harary, Consistency in marked digraphs, J. Math. Psych. 18 (1978) 260–269.
[5] L.W. Beineke, F. Harary, Consistent graphs with signed points, Riv. Mat. Sci. Econom. Social. 1 (1978) 81–88.
[6] F. Harary, On the notion of balance of a signed graph, Michigan Math. J. 2 (1954) 143–146.
[7] C. Hoede, A characterization of consistent marked graphs, J. Graph Theory 16 (1992) 17–23.
[8] J.E. Hopcroft, R.E. Tarjan, Dividing a graph into triconnected components, SIAM J. Comput. 2 (1973) 135–158.
[9] S.B. Rao, Characterization of harmonious marked graphs and consistent nets, J. Comb. Inf. Syst. Sci. 9 (1984) 97–112.
[10] F.S. Roberts, On the problem of consistent marking of a graph, Linear Algebra Appl. 217 (1995) 255–263.
[11] F.S. Roberts, On balanced signed graphs and consistent marked graphs, Electron. Notes Discrete Math. 2 (1999) 94–105.
[12] F.S. Roberts, S. Xu, Characterizations of consistent marked graphs, Discrete Appl. Math. 127 (2003) 357–371.
[13] T. Zaslavsky, Signed graphs, Discrete Appl. Math. 4 (1982) 47–74.
[14] T. Zaslavsky, A mathematical bibliography of signed and gain graphs and allied areas, in: Dynamic Surveys in Combinatorics, No. DS8, Electron. J. Combin. (1998).