Balanced group-labeled graphs

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ABSTRACT

A group-labeled graph is a graph whose vertices and edges have been assigned labels from some abelian group. The weight of a subgraph of a group-labeled graph is the sum of the labels of the vertices and edges in the subgraph. A group-labeled graph is said to be balanced if the weight of every cycle in the graph is zero. We give a characterization of balanced group-labeled graphs that generalizes the known characterizations of balanced signed graphs and consistent marked graphs. We count the number of distinct balanced labellings of a graph by a finite abelian group $\Gamma$ and show that this number depends only on the order of $\Gamma$ and not its structure. We show that all balanced labellings of a graph can be obtained from the all-zero labeling using simple operations. Finally, we give a new constructive characterization of consistent marked graphs and markable graphs, that is, graphs that have a consistent marking with at least one negative vertex.

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1. Introduction

The study of signed and marked graphs has a long history and diverse applications [11,14]. A signed graph is an undirected graph whose edges are labeled positive or negative. A marked graph is an undirected graph whose vertices are signed. Signed and marked graphs may be considered to be special cases of group-labeled graphs. In this paper, we generalize some results known for signed and marked graphs to group-labeled graphs. This generalization also gives some new properties of signed and marked graphs.

Let $\Gamma$ be a finite abelian group. A $\Gamma$-labeled graph is a graph whose vertices and edges are assigned labels from $\Gamma$. A $\Gamma$-labeled graph is denoted $(G, w)$, where $G$ is a graph and $w : V(G) \cup E(G) \to \Gamma$ is an assignment of labels from $\Gamma$ to the vertices and edges of $G$. Thus a signed graph may be considered to be a $\mathbb{Z}_2$-labeled graph with all vertex labels 0, while a marked graph has all edge labels 0.

One of the central notions in the study of signed graphs is that of balance. A signed graph is said to be balanced if every cycle in the graph contains an even number of edges with negative sign. Similarly, a marked graph is said to be consistent if every cycle contains an even number of vertices with negative sign. We generalize the notions of balance and consistency and their characterizations to group-labeled graphs.

Let $(G, w)$ be a group-labeled graph. The weight of a subgraph $H$ of $G$, denoted by $w(H)$, is $\sum_{x \in V(H) \cup E(H)} w(x)$. The graph $(G, w)$ is said to be a balanced group-labeled graph and $w$ is a balanced labeling of $G$ if $w(C) = 0$ for all cycles $C$ in $G$. Thus balanced $\mathbb{Z}_2$-labeled graphs include both balanced signed graphs and consistent marked graphs.

Balanced signed graphs were first characterized by Harary [6] who proved the following theorem.

Theorem 1. A signed graph $G$ is balanced iff there exists a subset $A \subseteq V(G)$ such that an edge has a negative sign iff it has exactly one end vertex in $A$. 

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Characterizations of consistent marked graphs were obtained much later. Initial results were given by Acharya and Rao [1,2,9] and a simpler characterization was obtained by Hoede [7], who proved the following theorem.

**Theorem 2.** A connected marked graph \( G \) is consistent iff there exists a spanning tree \( T \) in \( G \) such that

(i) every fundamental cycle with respect to \( T \) has an even number of vertices with negative sign;

(ii) any path in \( T \), which is the intersection of two fundamental cycles, has end vertices with the same sign.

Subsequently, Roberts and Xu [12] gave several alternative characterizations of consistent marked graphs that were similar to Hoede’s characterization.

Our main result in Section 2 is a generalization of Theorem 2 to group-labeled graphs. The characterizations of Roberts and Xu also extend to group-labeled graphs in a natural way. In Section 3, we give a simple formula for the number of balanced labelings of a graph by a finite abelian group \( \Gamma \). An interesting observation is that this number depends only on the order of \( \Gamma \) and not its structure.

Another characterization of balanced signed graphs is in terms of the notion of switching [13]. A switch applied to a vertex of a signed graph changes the sign of edges incident with it, keeping the signs of all other edges the same. A switch applied to any vertex of a balanced signed graph gives a balanced signed graph. It follows easily from Theorem 1 that a signed graph is balanced iff all edges can be made positive by switching a subset of vertices.

In Section 4, we define some simple operations on group-labeled graphs that preserve balance, and show that a group-labeled graph is balanced iff all edge and vertex labels can be made 0 using these operations. These operations generalize the switching operation for signed graphs.

Beineke and Harary [4] characterized directed graphs whose vertices can be consistently marked such that at least one vertex has a “−” sign, and in [5], they left open the same question for undirected graphs. Graphs that admit such a marking are called **markable**. Roberts [10] characterized 2-connected markable graphs in which the longest cycle has length at most five. In Section 5, we give a new constructive characterization of consistent marked graphs, which immediately gives a simple constructive characterization of markable graphs.

The notation and terminology used is largely standard, however, we clarify a few terms that are frequently required. The graphs that we consider are finite and undirected but may have loops and/or multiple edges. We assume that each edge has two end vertices that may possibly be the same, in which case it is a loop, else it is a link. The vertex set of a graph \( G \) is denoted by \( V(G) \) and the edge set by \( E(G) \).

A path \( P \) is a sequence of distinct vertices and edges \( v_0, e_1, v_1, \ldots, e_l, v_l \) such that the end vertices of \( e_i \) are \( v_{i-1} \) and \( v_i \) for \( 1 \leq i \leq l \). The path \( P \) is said to be a \( v_0-v_l \) path of length \( l \). The end vertices of \( P \) are \( v_0 \) and \( v_l \), all other vertices are internal. If \( A \subset V(G) \) and \( v \in V(G) \setminus A \), then an \( A-v \) path is a \( u-v \) path for some \( u \in A \), whose internal vertices are not in \( A \). A cycle \( C \) is a sequence of vertices and distinct edges \( v_0, e_1, v_1, \ldots, v_{l-1}, e_l, v_l \), such that \( v_0 = v_l \) and all other vertices are distinct, and \( e_i \) has end vertices \( v_{i-1} \) and \( v_i \), for \( 1 \leq i \leq l \).

A tree \( T \) in a graph \( G \) is a connected subgraph of \( G \) without cycles. If \( u, v \) are vertices in a tree \( T \), we denote by \( T[u, v] \) the unique path contained in \( T \) having end vertices \( u \) and \( v \). In particular, if \( P \) is a path and \( u, v \) are vertices in \( P \), then \( P[u, v] \) is the subpath of \( P \) with end vertices \( u, v \). If \( T \) is a spanning tree in a graph \( G \), a fundamental cycle with respect to \( T \) is a cycle in \( G \) containing exactly one edge not in \( T \). The edges not in \( T \) are called cotree edges. If \( A \subset V(G) \) is a non-empty proper subset of \( V \), then the set of edges having exactly one end vertex in \( A \) is called a cut in the graph.

If \( G_1, G_2 \) are graphs then the graph \( G_1 \cup G_2 \) has vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \). If \( G_1 \) is a subgraph of a graph \( G \) and \( uv \) is an edge in \( G \), let \( G_1 \cup uv \) be the graph with vertex set \( V(G_1) \cup u \cup v \) and edge set \( E(G_1) \cup \{ uv \} \).

In this paper, we will consider group-labeled graphs \( (G, w) \) whose vertices and edges are assigned labels from some arbitrary finite abelian group \( \Gamma \). However, if \( \Gamma \cong \Gamma_1 \times \Gamma_2 \) for some non-trivial groups \( \Gamma_1 \) and \( \Gamma_2 \), then a labeling of \( G \) by \( \Gamma \) may be considered to be a product of labelings by \( \Gamma_1 \) and \( \Gamma_2 \). The labeling by \( \Gamma \) is balanced iff the corresponding labelings by \( \Gamma_1 \) and \( \Gamma_2 \) are balanced, and the number of balanced labelings by \( \Gamma \) is the product of the number of balanced labelings by \( \Gamma_1 \) and \( \Gamma_2 \). Therefore, to understand balanced group-labeled graphs, it is sufficient to consider the case when \( \Gamma \) is a cyclic group, in particular \( \mathbb{Z}_k \) for some positive integer \( k \). Further, there is no loss of generality in assuming the graphs being considered are connected. Thus, in the rest of the paper, a group-labeled graph \( (G, w) \) will be a connected graph \( G \) whose vertices and edges have been assigned labels from \( \mathbb{Z}_k \) for some fixed positive integer \( k \).

We will assume elements of \( \mathbb{Z}_k \) to be \( \{0, 1, 2, \ldots, k - 1\} \). However, sometimes we need to consider the labels to be integers rather than elements of \( \mathbb{Z}_k \). In order to do this conveniently, we define the functions \( \pi : \mathbb{Z} \to \mathbb{Z}_k \) and \( \pi' : \mathbb{Z}_k \to \mathbb{Z} \), where \( \pi(i) = i \mod k \) for \( i \in \mathbb{Z} \) and \( \pi'(j) = j \) for \( j \in \mathbb{Z}_k \). Also, we will use \( \oplus \) (or \( \odot \)) to denote addition (or scalar multiplication) modulo \( k \) and \( + (-) \) to denote ordinary integer addition (or scalar multiplication).

**2. Characterization**

In this section, we generalize the characterizations of balanced signed graphs (Theorem 1) and consistent marked graphs (Theorem 2) to group-labeled graphs.

**Definition 1.** Let \( (G, w) \) be a group-labeled graph. An \( u-v \) path \( P \) in \( G \) is said to be **good** if \( 2w(P) = w(u) \oplus w(v) \). An edge \( uv \) is said to be **good** if the path of length 1 containing the edge \( uv \) is good, that is if \( 2w(uv) \oplus w(u) \oplus w(v) = 0 \). An edge or path that is not good is said to be **bad**.
Lemma 1. Let $(G, w)$ be a group-labeled graph. Let $P$ be a $u$–$v$ path in $G$ and let $x$ be any vertex in $P$. Suppose $P[u, x]$ is a good path. Then $P$ is good if $P[x, v]$ is good.

Proof. Since $P[u, x]$ is good, $2w(P[u, x]) = w(u) + w(x)$. Then $2w(P) = 2w(P[u, x]) + w(P[x, v]) + w(x) = 2w(P[x, v]) + w(u) + w(x)$. Therefore, $2w(P) = w(u) + w(x)$ if $2w(P[x, v]) = w(x) + w(v)$. \(\square\)

Lemma 2. Let $(G, w)$ be a group-labeled graph and let $P$ be any $u$–$v$ path in $G$. Let $x_1, y_1$ be two distinct vertices in $P$ such that $|P[u, x_1]| < |P[u, y_1]|$. Let $Q$ be an $x_1$–$y_1$ path in $G$ that is internally vertex-disjoint from $P$. Suppose $Q$ and $P[x_1, y_1]$ are good paths. Then $P$ is good if the path $P' = P[u, x_1] \cup Q \cup P[y_1, v]$ is good.

Proof. This follows from the fact that $2w(P) \oplus 2w(P') = 2w(P[x_1, y_1]) \oplus 2w(Q) = 0$, since $2w(P[x_1, y_1], y_1) = 2w(Q) = w(x_1) \oplus w(y_1)$. \(\square\)

Lemma 3. Let $(G, w)$ be a group-labeled graph in which all edges are good, that is, $2w(uv) \oplus w(u) \oplus w(v) = 0$ for all edges $uv$. Then $(G, w)$ is balanced if one of the following holds.

(i) $k$ is odd and for every edge $uv$, $w(uv) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u, v))/2$, where $\delta(u, v) = k$ if $\pi'(w(u)) + \pi'(w(v))$ is odd and $\delta(u, v) = 0$ if $\pi'(w(u)) + \pi'(w(v))$ is even.

(ii) $k$ is even and there exists a subset $A \subseteq V(G)$ such that for every edge $uv$, $w(uv) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u) + \delta(v))/2$, where $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$ if $x \notin A$, for all $x \in V(G)$.

Proof. First suppose $(G, w)$ is balanced. If $k$ is odd and every edge $uv$ is good, for any given values of $w(u)$ and $w(v)$, there is a unique value of $w(uv)$ that satisfies $2w(uv) \oplus w(u) \oplus w(v) = 0$. This value is exactly $\pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u, v))/2$.

Suppose $k$ is even. Since all edges are good, $w(u)$ and $w(v)$ must have the same parity for all edges $uv$. Since $G$ is connected, all vertex labels must have the same parity. Given $w(u)$ and $w(v)$ having the same parity, there are two possible values of $w(uv)$ that satisfy $2w(uv) \oplus w(u) \oplus w(v) = 0$: either $w(uv) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u, v))/2$ or $w(uv) = \pi(-\pi'(w(u)) + \pi'(w(v)) + k)/2$. Since every edge in $G$ is good, by Lemma 1, every path in $G$ is good. A good $u$–$v$ path $P$ is said to be small if $w(P) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u, v))/2$ and large if $w(P) = \pi((-\pi'(w(u)) + \pi'(w(v)) + k)/2$.

Let $T$ be any spanning tree in $G$ and let $r$ be any fixed vertex in $G$. Let $A \subseteq V(G)$ be the set of vertices $v$ such that $T[r, v]$ is small. We show that for any edge $uv$, $w(uv) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u) + \delta(v))/2$, where $\delta(x) = 0$ if $x \in A$ and $\delta(x) = k$ if $x \notin A$. Note that $r \in A$ and for any vertex $v$, $w(T[r, v]) = \pi((-\pi'(w(r)) + \pi'(w(v))) + \delta(v))/2$.

Let $uv$ be an edge not in $T$. Since $uv$ is a balanced labeling, $w(uv) \oplus w(T[u, v]) = 0$. Let $T[r, x] = T[r, u] \cap T[r, v]$. Then $w(T[u, v]) = w(T[r, u]) \oplus w(T[r, v]) \oplus 2w(T[r, x]) \oplus w(x) = \pi((-\pi'(w(r)) + \pi'(w(u)) + \delta(u))/2) \oplus \pi((-\pi'(w(r)) + \pi'(w(v)) + \delta(v))/2) \oplus \pi((-\pi'(w(v))) + \delta(v))/2.

Conversely, suppose $(G, w)$ is a group-labeled graph in which all edges are good. Suppose $k$ is odd, and for every edge $uv$, $w(uv) = \pi((-\pi'(w(u)) + \pi'(w(v))) + \delta(u, v))/2$. In any cycle $C$, there are an even number of edges $uv$ such that $\delta(u, v) = k$. Then the weight of any cycle $C = v_0, e_1, \ldots, e_l, v_l$ is

$$\pi \left( \sum_{1 \leq i \leq l} \pi'(w(v)) + \pi'(w(e_i)) \right) = \pi \left( \sum_{1 \leq i \leq l} \pi'(w(v)) + \pi'(w(v)) + \delta(v_{i-1}, v_i)/2 \right)$$

$$= \pi \left( \sum_{1 \leq i \leq l} \delta(v_{i-1}, v_i)/2 \right) = 0.$$

Thus $(G, w)$ is balanced.

Note that Lemma 3 can be considered to be a generalization of the characterization of balanced signed graphs, since if all vertex labels in a $\mathbb{Z}_2$-labeling are 0, then all edges are good.
Definition 2. A spanning tree $T$ in a group-labeled graph $(G, w)$ is good if
(i) every fundamental cycle with respect to $T$ has weight 0;  
(ii) any path in $T$, which is the intersection of two distinct fundamental cycles with respect to $T$, is good.

Lemma 4. Let $(G, w)$ be a group-labeled graph and suppose it contains a good spanning tree $T$. Let $P_1$, $P_2$, $P_3$ be 3 edge-disjoint paths between vertices $u$ and $v$ in $G$. Then $P_1$, $P_2$, $P_3$ are good paths in $G$.

Proof. Let $H$ be the graph $P_1 \cup P_2 \cup P_3$. The proof is by induction on the number of cotree edges in $H$, that is edges in $H$ that are not in $T$.

Case 1. Suppose there is a cotree edge $xy$ in $H$ such that the path $T[x, y]$ is not contained in $H$. Then we can find two distinct vertices $x_1$, $y_1$ in $T[x, y]$ such that $V(T[x_1, y_1]) \cap V(H) = \{x_1, y_1\} \cap E(H) = \emptyset$.

Case 2. Suppose the paths are internally vertex-disjoint.

Case 2.1. Suppose there is a cotree edge $xy$ in $H$ such that the path $T[x, y]$ is contained in $H$. Then we can find two distinct vertices $x_1$, $y_1$ in $T[x, y]$ such that $V(T[x_1, y_1]) \cap V(H) = \{x_1, y_1\} \cap E(H) = \emptyset$.

Theorem 3. A connected group-labeled graph $(G, w)$ is balanced if and only if it contains a good spanning tree. If $(G, w)$ contains a good spanning tree, then all spanning trees in $(G, w)$ are good.
If there is only one cotree edge in $C$, then $C$ is a fundamental cycle with respect to $T$ and has weight 0, since $T$ is good.

If there are at least two cotree edges in $C$, then there exist two distinct vertices $x, y \in V(C)$ such that $V(T[x, y]) \cap V(C) = \{x, y\}$ and $E(T[x, y]) \cap E(C) = \emptyset$. Let $P_1, P_2$ be the two edge-disjoint $x$-$y$ paths in $C$. Then, by Lemma 4, $P_1, P_2, T[x, y]$ are good paths. Also, each of $P_1, P_2$ contains at least one cotree edge. Then, by induction, the cycles $C_1 = P_1 \cup T[x, y]$ and $C_2 = P_2 \cup T[x, y]$ have weight 0. Hence $w(C) = w(P_1) + w(P_2) + w(x) + w(y) = w(C_1) + w(T[x, y]) + w(x) + w(y) + w(C_2) + w(T[x, y]) \cap w(x) + w(y) = 0$. Thus $C$ also has weight 0. Hence $(G, w)$ is balanced and every spanning tree in $G$ is good. □

Note that from Lemma 4 and Theorem 3, we can conclude the following corollary.

**Corollary 1.** If $(G, w)$ is a balanced group-labeled graph and $P_1, P_2, P_3$ are 3 edge-disjoint $u$-$v$ paths in $G$, then $P_1, P_2, P_3$ are good paths.

The characterizations given by Roberts and Xu [12] can also be generalized in a similar way. Since the proofs are almost the same as in the case of marked graphs, we mention only one, which follows immediately from Theorem 3.

**Corollary 2.** A connected group-labeled graph $(G, w)$ is balanced iff there is a spanning tree $T$ in $G$ such that all cycles containing at most two cotree edges have weight 0.

### 3. Counting

In this section, we give a simple formula for the number of distinct balanced labellings of a graph $G$ by $Z_k$, and hence the number of balanced labellings of $G$ by an arbitrary finite abelian group $F$.

**Definition 3.** A 3-edge-connected component of a graph $G$ is a maximal subset of vertices of $G$, such that for any two vertices in the subset there are 3 edge-disjoint paths between them in $G$.

Note that the existence of 3 edge-disjoint paths between a pair of vertices is an equivalence relation on the set of vertices, and the 3-edge-connected components of the graph are the equivalence classes of this relation. Thus distinct 3-edge-connected components of a graph are disjoint and partition the vertex set. Let $c_3(G)$ denote the number of 3-edge-connected components of a graph $G$. Let $W(G, k)$ denote the set of balanced labellings of $G$ by $Z_k$ and let $w(G, k) = |W(G, k)|$.

**Theorem 4.** For any connected graph $G$ and positive integer $k$

$$w(G, k) = k^{c_3(G)} - 1.$$

**Proof.** The proof is by induction on $|G|$.

**Case 1.** Suppose there is cut $X$ in $G$ of size 1. Let $G_1$ and $G_2$ be the components of $G \setminus X$. Then a balanced labeling of $G$ is obtained from balanced labellings of $G_1$ and $G_2$ by assigning an arbitrary label to the edge in $X$. Thus $w(G, k) = w(G_1, k) \times w(G_2, k) \times k$. Since $|G| = |G_1| + |G_2|$ and $c_3(G) = c_3(G_1) + c_3(G_2)$, the theorem follows by induction.

**Case 2.** Suppose $G$ is 2-edge-connected and has a 2-edge-cut $X$. Let $G_1$ and $G_2$ be the components of $G \setminus X$. Let $X = \{e_1, e_2\}$ such that $e_1, e_2 \in V(G)$ for $i \in [1, 2]$. Let $p_i, q_i$ be the same vertex as $q_i$. Let $G'_i$ be the graph obtained from $G_i$ by adding a new edge $e_i$ with vertices $p_i, q_i$ for $i \in [1, 2]$. Then $|G| = |G'_1| + |G'_2|$ and $c_3(G) = c_3(G'_1) + c_3(G'_2)$. It is sufficient to show that $w(G, k) = w(G'_1, k) \times w(G'_2, k) \times k$. The theorem then follows by induction.

We show a bijection $F : W(G, k) \rightarrow W(G'_1, k) \times W(G'_2, k) \times Z_k$. Let $w$ be any balanced labeling of $G$ by $Z_k$. Let $w_{e_i}$ be the labeling of $G'_i$ defined by $w_{e_i}(x) = w(x)$ for $x \in V(G'_i) \cup E(G'_i)$ and $w_{e_i}(e_i) = w(P_{e_{i-1}}) + w(p_iq_i) + w(P_{e_{i+1}})$, where $P_{e_i}$ is any $p_iq_i$-path in $G_i$. Note that since $w$ is balanced, all $p_iq_i$-paths in $G_i$ must have the same weight, so $w_{e_i}$ is well-defined. This also implies that $w_{e_i}$ is a balanced labeling of $G'_i$. Define $F(w) = (w_{e_1}, w_{e_2}, w(p_1q_2))$.

Conversely, suppose $w_{e_1}$ and $w_{e_2}$ are balanced labellings of $G'_1$ and $G'_2$, respectively, and let $a \in Z_k$. We show that there is a unique balanced labeling $w$ such that $F(w) = (w_{e_1}, w_{e_2}, a)$. Define $w(x) = w_{e_1}(x)$ for all $x \in V(G'_1) \cup E(G'_1)$ and $w(p_1q_2) = a$. Let $w(q_1q_2) = w_{e_1}(e_1) \oplus w_{e_2}(e_2) \oplus a$. It is then easy to check that $w$ is a balanced labeling of $G$ and $F(w) = (w_{e_1}, w_{e_2}, a)$.

**Case 3.** Suppose $G$ is 3-edge-connected and let $w$ be any balanced labeling of $G$ by $Z_k$. For every edge $wuv$ in $G$, there are 2 edge-disjoint $u$-$v$ paths in $G - wuv$, hence Corollary 4 implies all edges are good, and $2w(uv) = w(u) + w(v) = 0$.

By Lemma 3, if $k$ is odd, then for any edge $wuv$, $w(uv) = \pi(-\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2$, where $\delta(u, v) = 0$ if $\pi'(w(u)) + \pi'(w(v))$ is even and $\pi'(w(u)) + \delta(u, v)$, otherwise. Since the labels of the vertices can be arbitrary, the number of distinct balanced labellings is $k^{c_3(G)}$. Since $c_3(G) = 1$, the theorem holds.

If $k$ is even, all vertex labels must have the same parity. Again by Lemma 3, there exists a subset $A$ of vertices such that for any edge $wuv$, $w(uv) = \pi(-\pi'(w(u)) + \pi'(w(v)) + \delta(u, v))/2$, where $\delta(u, v) = 0$ if $x \in A$ and $\delta(u, v) = k$, otherwise. The labeling obtained is the same if the set $A$ is replaced by $V(G) \setminus A$. Thus the total number of balanced labellings is $2k^{c_3(G)} - 2^{c_3(G) - 1} = k^{c_3(G)}$.

This completes the proof of Theorem 4. □

The following corollary of Theorem 4 follows immediately by induction.
Corollary 3. The number of distinct balanced labellings of a connected graph $G$ by an arbitrary finite abelian group $\Gamma'$ is $|\mathcal{L}(G, \Gamma')| = |\mathcal{L}(G, \mathbb{Z})|^{|G| + c_2(G) - 1|} - 1$.

It may be noted that in [3], edge weightings by real numbers in which all cycles have total weight 0 were considered. It was shown that they form a vector space of dimension $c_2(G) - 1$. If vertex labels are also allowed, the dimension becomes $|G| + c_2(G) - 1$. This can be proved in the same way as Theorem 4.

4. Switching

In this section, we define a few simple operations on group-labeled graphs, called shifting, that preserve balance, and show that any balanced labeling of a graph can be obtained from the all-zero labeling using these operations. Let $(G, 0)$ denote the group-labeled graph $G$ with all vertex and edge labels 0.

Definition 4. We define the following operations on a group-labeled graph $(G, w)$. Here $a$ denotes an arbitrary element of $\mathbb{Z}_k$.

(i) Let $u$ be any vertex in $G$. Let $(G, w')$ be the group-labeled graph defined by $w'(u) = w(u) + 2a$, $w'(e) = w(e) \oplus 2a$ if $e$ is a loop incident with $u$, and $w'(e) = w(e) \oplus a$ if $e$ is a link incident with $u$, and $w'(x) = w(x)$ for all other vertices and edges $x$ in $G$. $(G, w')$ is said to be obtained from $(G, w)$ by shifting the vertex $u$ by $a$.

(ii) Let $X$ be a minimal cut of size at most two in $G$. Let $e_1$ be an edge in $X$ and let $G'$ be a component of $G \setminus X$. Let $(G, w)$ be the group-labeled graph defined by $w'(v) = w(v) \oplus a$ for all vertices $v \in V(G_1)$, $w'(e) = w(e) \oplus a$ for all edges $e \in E(G_1) \cup \{e_1\}$ and $w'(x) = w(x)$ for all other vertices and edges $x$ in $G$. $(G, w')$ is said to be obtained from $(G, w)$ by shifting the edge $e_1$ in the cut $X$ by $a$.

(iii) Let $(G, w)$ be the group-labeled graph defined by $w'(v) = w(v) \oplus a$ for all vertices $v \in V(G)$ and $w'(e) = w(e) \oplus a$ for all edges $e \in E(G)$. $(G, w')$ is said to be obtained from $(G, w)$ by shifting the graph $G$ by $a$.

Note that if $k = 2$, then switching a vertex is equivalent to shifting it by 1. A group-labeled graph $(G, w)$ is said to be shift equivalent to the graph $(G, w')$ if $(G, w')$ can be obtained from $(G, w)$ by a sequence of shift operations.

Theorem 5. A group-labeled graph $(G, w)$ is balanced iff it is shift equivalent to $(G, 0)$.

Proof. It is easy to verify that each of the shifting operations preserves balance. Thus any group-labeled graph $(G, w)$ that is shift equivalent to $(G, 0)$ is balanced. To prove the converse, suppose $(G, w)$ is a balanced group-labeled graph.

Let $(G, w')$ be a group-labeled graph shift equivalent to $(G, w)$ such that the order of the largest connected component of the spanning subgraph of $(G, w')$ consisting of good edges is as large as possible and let $H$ be this component.

Case 1. Suppose $H$ spans $G$. We claim that all edges in $(G, w')$ are good. If $uv$ is a bad edge, let $P$ be a $u$-$v$ path in $H$ consisting of good edges. Since the cycle $P \cup uv$ has weight 0, $2w(uv) + 2w(P) = 0$ and since $P$ is a good path, $2w(P) = w(u) + w(v)$. This implies $w(u)$ is a good edge, a contradiction.

Thus, $(G, 0)$ can be obtained from $(G, w')$ by shifting each vertex $x$ by $\pi(-(\pi'(w'(x))) + \delta(x))$, where for $x \in V(G)$, $\delta(x) = 0$ if $x$ is a vertex of $A$ and $\delta(x)$ is 1 otherwise. If $\pi'(w'(x))$ is even for all vertices $x$, then $(G, 0)$ can be obtained from $(G, w')$ by shifting vertex $x$ by $\pi(-(\pi'(w'(x))) + \delta(x))$ for all vertices $x$. If $\pi'(w'(x))$ is odd for all vertices $x$, then $(G, 0)$ can be obtained from $(G, w')$ by first shifting the graph $G$ by 1 and then shifting vertex $x$ by $\pi(-(\pi'(w'(x))) + \delta(x))/2$ for all vertices $x$. This implies $(G, w)$ is shift equivalent to $(G, 0)$.

Case 2. Suppose $V(H) \subseteq V(G)$. Since $G$ is connected, there exists an edge $uv$ in $G$ such that $u \in V(H)$ and $v \not\in V(H)$. Clearly, $uv$ is a bad edge. We claim that $G - uv$ does not contain 2 edge-disjoint $H$-$v$ paths. Suppose there exist two such paths $P_1, P_2$ whose end vertices in $V(H)$ are $u_1$ and $u_2$ (not necessarily distinct). Then there is a vertex $x \in V(H)$ such that $H$ contains at least 3 edge-disjoint paths $Q_1, Q_2, Q_3$ such that $Q_i$ contains an $x-u_i$ path for $i \in \{1, 2\}$ and $Q_3$ contains an $x-u$ path. The vertex $x$ may possibly be one of $[u_1, u_2, u]$. Then $Q_3 \cup uv, Q_3 \cup P_1, Q_3 \cup P_2$ are 3 edge-disjoint $x$-$v$ paths in $G$. By Corollary 1, $Q_3 \cup uv$ is a good path, and since $Q_3$ is also good, by Lemma 1, the edge $uv$ is also good, a contradiction.

Therefore, there exists a minimal cut $X$ of size at most two that separates $V(H)$ and $v$ and $H$ is a subgraph of some component of $G - X$. Let $(G, w')$ be the group-labeled graph obtained from $(G, w)$ by shifting the edge $uv$ contained in the cut $X$ by $2w'(uv) \oplus w'(u) \oplus w'(v)$. Then all edges in $H$ as well as the edge $uv$ are good in $(G, w')$. But this contradicts the choice of the graph $(G, w')$ and the component $H$.

This completes the proof of Theorem 5. □

5. Markable graphs

In this section, we give a new constructive characterization of consistent marked graphs. This immediately gives a characterization of markable graphs, that is, graphs for which there exists a consistent marking of the vertices with at least one vertex having a ‘−’ sign.
Proof. It is easy to verify that a marked graph satisfying any one of the conditions in Theorem 6 is consistent.

Now we prove the converse. Suppose G is a 2-edge-connected graph and let w be any consistent marking of G.

Suppose there is a 2-edge-cut X = {p₁, p₂, q₁, q₂} in G. Let G₁, G₂ be the components of G − X such that {p₁, q₁} ⊆ V(G₁).

Since w is a consistent marking of G, the parity of the number of negative vertices on all p₁−q₁ paths in G₁ and p₂−q₂ paths in G₂ must be the same.

Suppose all p₁−q₁ paths in G₁ have even number of negative vertices. Let G’ be the graph obtained from G₁ by adding an edge eᵢ with end vertices pᵢ and qᵢ. Then G’ is 2-edge-connected, and the marking wᵢ of G’ obtained by restricting w to V(Gᵢ) is a consistent marking of G’. Thus G satisfies property (iv) in Theorem 6.

Suppose all p₁−q₁ paths in G₁ have odd number of negative vertices. Suppose both G₁ and G₂ contain at least two vertices. Let G’₁ and G’₂ be the graphs obtained from G by contracting the subgraphs G₂ and G₁, and let v₂ and v₁ be the vertices representing the subgraphs G₂ and G₁ in G’₁ and G’₂, respectively. Then the marking wᵢ of G’₁ defined as wᵢ(v) = w(v) for all v ∈ V(G₁) and wᵢ(v₂) = ‘−’ is a consistent marking of G’. Similarly, wᵢ(v) = w(v) for all v ∈ V(G₂) and wᵢ(v₁) = ‘+’ is a consistent marking of G’. Since v₁, v₂ have degree 2 in G’₁ and G’₂, G satisfies property (v) in Theorem 6.

We may now assume that if there is a 2-edge-cut X in G, at least one component of G − X is a simple negative vertex. This also implies that G does not contain a positive vertex of degree 2.

Suppose two negative vertices of degree 2 are adjacent in G. If |G| > 3 then we get a 2-edge-cut X in G such that each component of G − X contains at least two vertices, a contradiction. If |G| ≤ 3 then either G contains 2 negative vertices joined by two edges or G is K₃ with two negative and one positive vertex. In the first case, G satisfies property (ii) in Theorem 6, while in the second, there is a positive vertex of degree 2, a contradiction.

So we may assume that no two negative vertices of degree 2 are adjacent. Let G’ be the graph obtained from G by replacing every negative vertex v of degree 2 by an edge having the neighbors of v as end vertices (the two neighbors of v need not be distinct). Then G’ is 3-edge-connected.

Let (G’, w’) be the Z₂-labeled graph, defined by w’(v) = 0 if w(v) = ‘+’ and w’(v) = 1 if w(v) = ‘−’ for all v ∈ V(G’), and w’(e) = 0 for edges e ∈ E(G) and w’(e) = 1 for the added edges, that is, for e ∈ E(G’) \ E(G). Then (G’, w’) is a balanced 3-edge-connected Z₂-labeled graph. By Corollary 1, all edges are good and hence all vertex labels have the same parity.

Suppose all vertex labels are 0. Then by Lemma 3, there exists a subset A ⊆ V(G’) such that w’(uv) = 1 iff exactly one of u, v is in A. If A = ∅ or A = V(G’), then there is no edge labeled 1, and G is a 3-edge-connected graph with all vertices positive. Thus G satisfies property (i) in Theorem 6. If ∅ ⊂ A ⊂ V(G’), then G satisfies property (iii) in Theorem 6.

If all vertex labels are 1, then there exists a subset A ⊆ V(G’) such that w’(uv) = 0 iff exactly one of u, v is in A. In this case, G is a bipartite graph with all vertices negative, and it satisfies property (ii) in Theorem 6. □

Theorem 6 leads to a linear-time algorithm to test whether a marked graph is consistent. This follows from the linear-time algorithm for finding the triconnected components of a graph [8], with a few simple modifications. As a corollary of Theorem 6, we get a constructive characterization of all markable 2-edge-connected graphs.

Corollary 4. A 2-edge-connected graph G is markable iff it satisfies one of the following properties.

(i) G is bipartite.
(ii) G is obtained from a 3-edge-connected graph G’ by subdividing exactly once all edges in some cut of G’.
(iii) G is obtained from the disjoint union of a markable 2-edge-connected graph G₁ and an arbitrary 2-edge-connected graph G₂, by replacing edges p₁q₁ ∈ E(G₁) and p₂q₂ ∈ E(G₂), by edges p₁p₂ and q₁q₂.
(iv) G is obtained from the disjoint union of two markable 2-edge-connected graphs G₁ and G₂, by deleting a vertex pᵢ in Gᵢ of degree 2, and adding edges q₁q₂ and rᵢr₂, where qᵢ, rᵢ are the neighbors of pᵢ in Gᵢ, for i ∈ {1, 2}.

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