# Agent Failures in Totally Balanced Games and Convex Games

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**Abstract.** We examine the impact of independent agents failures on the solutions of cooperative games, focusing on totally balanced games and the more specific subclass of convex games. We follow the reliability extension model, recently proposed in [1] and show that a (approximately) totally balanced (or convex) game remains (approximately) totally balanced (or convex) when independent agent failures are introduced or when the failure probabilities increase. One implication of these results is that any reliability extension of a totally balanced game has a non-empty core. We propose an algorithm to compute such a core imputation with high probability. We conclude by outlining the effect of failures on non-emptiness of the core in cooperative games, especially in totally balanced games and simple games, thereby extending observations in [1].

**Keywords:** Totally Balanced Games, Convex Games, Agent Failures, Cooperative Game Theory

## 1 Introduction

Consider a communication network designed to transmit information from a source node to a target node, where selfish agents control the different links in the network. Suppose the utility generated by the network is proportional to the bandwidth it can achieve between the source and the target. Further suppose that the links are not fully reliable and may fail, and that these link failures are independent of each other, although the failure probability of each link may be different. The surviving links provide a certain bandwidth from the source to the target. Since it is not known a priori which links would fail, there is uncertainty regarding the revenue that the agents would generate.

In such a setting the agents owning the links typically need each other in order to generate revenue, but since they are selfish each of them attempts to maximize his own share of the revenue. Which agreements are these agents likely to make regarding sharing the revenue? How do the link failures affect the agents' ability to reach a stable agreement regarding distributing the gains amongst themselves? Can we compute such stable payment allocations?

Interactions between selfish agents who must cooperate to achieve their goals are analyzed using cooperative game theory [18,7], where solution concepts attempt to characterize how such agents might agree to act and share the resulting gains among themselves. A prominent such concept is the core [11] which requires that no sub-coalition could defect and improve its utility by operating on its own.

Shapley and Shubik introduced the class of *totally balanced* games and showed that such games have a non-empty core [24]. Many interesting and practical classes of games have been shown to be totally balanced, such as the network flow game [13], Owen's linear production game [19], the permutation game [27], the assignment game [25], the minimum cost spanning tree game [12], etc. Our motivating example is a network flow game with independent agent failures.

Despite the wide coverage of cooperative interactions, most models ignore failures although it is hardly realistic to assume that all agents can always fill their roles. We use the reliability extension model [1] which formalizes independent agent failures in cooperative games, to investigate such failures in totally balanced games and in the more specific subclass of convex games.

**Our Contribution:** We first study the reliability extension of general cooperative games. We show how a game is transformed when failures are introduced or when the reliabilities (probabilities of agents not failing) change. Next we introduce the class of $\epsilon$-totally balanced games, a natural generalization of totally balanced games, and investigate the effect of failures on such games. We prove that every reliability extension of an $\epsilon$-totally balanced game is also $\epsilon$-totally balanced. For $\epsilon = 0$, this implies that every reliability extension of a totally balanced game is also totally balanced. Further, we show that decreasing one or more reliabilities in an $\epsilon$-totally balanced game keeps it $\epsilon$-totally balanced.

Using Shapley's result that convex games are totally balanced [23], our results imply that every reliability extension of a convex game is also totally balanced. This strengthens a result by Bachrach et. al. [1] who prove that every reliability extension of a convex game has a non-empty core. Similarly to $\epsilon$-totally balanced games, we also introduce $\epsilon$-convex games and prove that every reliability extension of an $\epsilon$-convex game is also $\epsilon$-convex. For $\epsilon = 0$, this implies that every reliability extension of a convex game is not just totally balanced, but convex. Further, we show that decreasing reliabilities in an $\epsilon$-convex game keeps it $\epsilon$-convex. We then prove that any $\epsilon/(n-1)$-convex game is $\epsilon$-totally balanced, generalizing Shapley's result that convex games are totally balanced. Additionally, we examine the computational aspects of a game's reliability extension, and provide an algorithm that computes a core solution of any reliability extension of a totally balanced game with high probability.

Our results show that both introducing failures and increasing failure probabilities preserve core non-emptiness in totally balanced games. We point out that neither of these preserve core non-emptiness in general cooperative games. Bachrach et. al. [1] observe that introducing failures preserves non-emptiness of the core in simple games (where every coalition has value either 0 or 1). Surprisingly, we show that this is not the case for increasing failure probabilities.

## 2 Related Work

Shapley and Shubik [24] introduced the notion of totally balanced games and proved their equivalence to the class of market games. Kalai and Zemel [13] later proved that they are also equivalent to two other classes of games: finite collections of simple additive games and network flow games. Owen [19] and Tijs et. al. [27] introduced two practical classes of totally balanced games: respectively, linear production games arising from linear programming problems and permutation games arising in sequencing and assignment problems. Deng et. al. [10] extended the analysis of total balancedness to various combinatorial optimization games, partition games and packing and convering games. These results suggest that the class of totally balanced games is elementary and practical.

Our analysis follows the reliability extension model of Bachrach et. al. [1] to examine the impact of independent agent failures in totally balanced games. A somewhat reminiscent model was proposed by Chalkiadakis et. al. [8] in which they consider the problem of coalition formation in a Bayesian setting. In their model, agents have types which are private information and agents have beliefs about the types of the other agents. In our setting, the failure probabilities can be viewed as types, but we focus on the specific case when these failure probabilities are public information and the failures are independent. Agent failures have also been widely studied in non-cooperative game theory. For example, Penn et. al. [20] study independent agent failures in congestion games, which are non-cooperative normal form games. Such failures have also been studied in other fields such as reliable network formation [4], non-cooperative Nash networks [5], sensor networks [14] etc. It is quite surprising that such an elementary notion of failure was only recently formalized in cooperative games.

## 3 Preliminaries

A transferable utility cooperative game $G = (N, v)$ is composed of a set of agents $N = \{1, 2, \dots, n\}$ and a characteristic function $v : 2^N \to \mathbb{R}$ indicating the total utilities achievable by various coalitions (subsets of agents). By convention, $v(\emptyset) = 0$. For any agent $i \in N$ and coalition $S \subseteq N$, we denote $S \cup \{i\}$ by $S + i$ and $S \setminus \{i\}$ by $S - i$. For a game $G = (N, v)$ and coalition $S \subseteq N$, $G_S$ denotes the subgame of $G$ obtained by restricting the set of agents to $S$.

**Convex Games:** A characteristic function is called *supermodular* if for each $i \in N$ and for all $S$ and $T$ such that $S \subseteq T \subseteq N - i$, we have $v(S+i) - v(S) \leq v(T+i) - v(T)$ (i.e., increasing marginal returns). A game is called *convex* if its characteristic function is supermodular. Similarly, for any $\epsilon \geq 0$, a characteristic function is called $\epsilon$-supermodular if for each $i \in N$ and for all $S$ and $T$ such that $S \subseteq T \subseteq N - i$, we have $v(S+i) - v(S) \leq v(T+i) - v(T) + \epsilon$. Define a game to be $\epsilon$-convex if its characteristic function is $\epsilon$-supermodular. Note that convex games are recovered as the special case of $\epsilon = 0$.

**Imputation:** The characteristic function defines the value that a coalition can achieve on its own, but not how it should *distribute* the value among its members. A payment vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ is called a *pre-imputation* if $\sum_{i=1}^{n} p_i = v(N)$. A payment vector $\boldsymbol{p} = (p_1, \ldots, p_n)$ is called an imputation if it is a pre-imputation and also individually rational, i.e., $p_i \geq v(\{i\})$ for every $i \in N$. Here, $p_i$ is the payoff of agent $i$, and the payoff of a coalition $C$ is $p(C) = \sum_{i \in C} p_i$.

**Core:** A basic requirement for any good imputation is that the payoff to every coalition is at least as much it can gain on its own so that no coalition can gain by defecting. The *core* is the set of all imputations $\boldsymbol{p}$ such that $p(N) = v(N)$ and $p(S) \geq v(S)$ for all $S \subseteq N$. It may be empty or may contain more than one imputation. One closely related concept is that of $\epsilon$-core. For any $\epsilon \in \mathbb{R}$, the $\epsilon$-core is the set of all imputations $\boldsymbol{p}$ such that $p(N) = v(N)$ and for every $S \subseteq N$ such that $S \neq N$, we have $p(S) \geq v(S) - \epsilon$. When $\epsilon > 0$, it serves as a relaxation of the core and is useful in predicting behaviour in games where the core is empty. When $\epsilon < 0$, it serves as a stronger concept where every coalition requires at least an incentive of $|\epsilon|$ to defect. We denote the case of $\epsilon > 0$ as the approximate core and the case of $\epsilon < 0$ as the superstable core. For any game, it is easy to show that the set $\{\epsilon \mid \text{the } \epsilon\text{-core is non-empty}\}$ is compact and thus has a minimum element $\epsilon_{\min}$. The $\epsilon_{\min}$ is known as the *least core value* of the game and the $\epsilon_{\min}$-core is known as the *least core*.

**Total Balancedness:** As defined by Shapley and Shubik [24], a game is called *totally balanced* if every subgame of the game has a non-empty core. We define a natural generalization of totally balanced games. For any $\epsilon \geq 0$, a game is called *$\epsilon$-totally balanced* if every subgame of the game has non-empty $\epsilon$-core.

**Reliability Game:** As defined in [1], a *reliability game* $G = (N, v, \boldsymbol{r})$ consists of the set of agents $N = \{1, 2, \ldots, n\}$, the *base characteristic function* $v$ which describes the values of the coalitions in the absence of failures, and the reliability vector $\boldsymbol{r}$ where $r_i$ is the probability of agent $i$ not failing. After taking failures into account, the characteristic function of the reliability game, denoted by $v^{\boldsymbol{r}}$, is given by the following equation. For every coalition $S \subseteq N$,

$$v^{\boldsymbol{r}}(S) = \sum_{S' \subseteq S} \Pr(S'|S) \cdot v(S') = \sum_{S' \subseteq S} \left( \prod_{i \in S'} r_i \cdot \prod_{j \in S \setminus S'} (1 - r_j) \right) \cdot v(S'). \qquad (1)$$

Here, $\Pr(S'|S)$ denotes the probability that every agent in $S'$ survives and every agent in $S \setminus S'$ fails so $v^{\boldsymbol{r}}(S)$ is the expected utility $S$ achieves under failures. The set $S'$ is called the *survivor set* for the coalition $S$. For the *base game* $G = (N, v)$, the game $G^{\boldsymbol{r}} = (N, v, \boldsymbol{r})$ is called the *reliability extension* of $G$ with the reliability vector $\boldsymbol{r}$. For a reliability vector $\boldsymbol{r}$, we denote by $\boldsymbol{r_{\text{-}i}}$ the vector of reliabilities of all agents except $i$ and by $\boldsymbol{r'} = (p, \boldsymbol{r_{\text{-}i}})$ the reliability vector where $r'_i = p$ and $r'_j = r_j$ for $j \neq i$. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, define $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for every $i$.

## 4 Reliabilities, Total Balancedness and Convexity

We examine how the value of a coalition changes as the reliability of an agent changes in a general game.

**Lemma 1.** *Let $G = (N, v, \boldsymbol{r})$ be a reliability game. Let $i \in N$ be an agent and let $p = r_i > 0$ be the reliability of agent $i$ in $G$. Take $0 \le p' \le 1$ and let $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ be the characteristic functions of $G$ and $G'$ respectively. Then the following holds.*[3]

1. *For any coalition $S \subseteq N$ such that $i \notin S$, we have $v^{\boldsymbol{r'}}(S) = v^{\boldsymbol{r}}(S)$.*
2. *For any coalition $S \subseteq N$ such that $i \in S$, we have*

$$v^{\boldsymbol{r'}}(S) = \frac{p'}{p} \cdot v^{\boldsymbol{r}}(S) + \left(1 - \frac{p'}{p}\right) \cdot v^{\boldsymbol{r}}(S - i).$$

*Proof Sketch.* Part 1 of the proof is trivial and follows directly from Equation (1). For the second part, for any coalition $S \subseteq N$ such that $i \in S$, we define $v_i^{rel}(S)$ as the value of $S$ in the game $G_i^{rel} = (N, v, \boldsymbol{r_i^{rel}})$ where $\boldsymbol{r_i^{rel}} = (1, \boldsymbol{r_{-i}})$. Now we break the summation in Equation (1) into two parts: summation over the subsets containing $i$ and summation over the subsets not containing $i$, and observe that $v^r(S) = p \cdot v_i^{rel}(S) + (1 - p) \cdot v^{\boldsymbol{r}}(S - i)$ and similarly, $v^{\boldsymbol{r'}}(S) = p' \cdot v_i^{rel}(S) + (1 - p') \cdot v^{\boldsymbol{r'}}(S - i)$. Finally, observing that $v^{\boldsymbol{r'}}(S - i) = v^{\boldsymbol{r}}(S - i)$ (using part 1) and eliminating $v_i^{rel}(S)$ from the two equations, we get the desired result. ■

The proof appears in the full version of the paper.[4] Note that by starting with $G = (N, v, \mathbf{1})$ where $\mathbf{1} = \langle 1, 1, \ldots, 1 \rangle$, we can use Lemma 1 to analyze the effect of introducing failures into a cooperative game as well.

### 4.1 Approximately Totally Balanced Games

We now analyze the reliability extension of $\epsilon$-totally balanced games and prove that $\epsilon$-total balancedness is preserved when the reliability of one agent decreases.

**Theorem 1.** *Let $\epsilon \ge 0$ and $G = (N, v, \boldsymbol{r})$ be a reliability game that is $\epsilon$-totally balanced. Fix $i \in N$ and let $p = r_i > 0$ be the reliability of agent $i$ in $G$. Take $p'$ such that $0 \le p' \le p$ and define $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Then $G'$ is $\epsilon$-totally balanced.*

*Proof.* Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ denote the characteristic functions of $G$ and $G'$ respectively. We want to prove that $G'$ is $\epsilon$-totally balanced, i.e., for any coalition

[3] The equation in part 2 of Lemma 1 really captures both parts 1 and 2. Part 1 is obtained by observing that $S - i = S$ when $i \notin S$. The two cases are separated for clarity and for the convenience of the reader.

[4] The full version is available from: `http://www.cs.cmu.edu/~nkshah/papers.html`.

$S \subseteq N$, the subgame $G'_S$ has an $\epsilon$-core imputation. Fix any coalition $S \subseteq N$. There are two cases: $i \in S$ and $i \notin S$.[5]

If $i \notin S$, then using Lemma 1 we see that $v^{r'}(C) = v^r(C)$ for every $C \subseteq S$, i.e., the subgames $G'_S$ and $G_S$ are equivalent. Further, since $G$ is an $\epsilon$-totally balanced game, its subgame $G_S$ has an $\epsilon$-core imputation $\boldsymbol{x}$. It is easy to see that $\boldsymbol{x}$ is also an $\epsilon$-core imputation of $G'_S$.

Now let $i \in S$. Since $G$ is an $\epsilon$-totally balanced game, both its subgames $G_S$ and $G_{S-i}$ have $\epsilon$-core imputations, say $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ (take $\boldsymbol{x_\phi} = \langle 0, 0, \ldots, 0 \rangle$). Extend both vectors by setting the payments to agents in $N \setminus S$ (and payment to $i$ in $\boldsymbol{x_{S-i}}$) to be zero. We prove that $\boldsymbol{x} = {}^{p'}\!/_{p} \cdot \boldsymbol{x_S} + (1 - {}^{p'}\!/_{p}) \cdot \boldsymbol{x_{S-i}}$ is an $\epsilon$-core imputation of $G'_S$. First we show that $\boldsymbol{x}$ is a pre-imputation.

$$\begin{aligned} x(S) &= \frac{p'}{p} \cdot x_S(S) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(S) = \frac{p'}{p} \cdot x_S(S) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(S-i) \\ &= \frac{p'}{p} \cdot v^r(S) + \left[1 - \frac{p'}{p}\right] \cdot v^r(S-i) = v^{r'}(S), \end{aligned}$$

Here, the second transition follows since the payment to agent $i$ in $\boldsymbol{x_{S-i}}$ is 0, the third transition follows since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are $\epsilon$-core imputations of $G_S$ and $G_{S-i}$ respectively and the last transition follows from Lemma 1.

Now for any coalition $C \subseteq S$ and $C \neq S$, we want to show that $x(C) \geq v^{r'}(C) - \epsilon$. We again take two cases: $i \in C$ and $i \notin C$. Let $i \notin C$, i.e., $C \subseteq S - i$. Since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are $\epsilon$-core imputations of $G_S$ and $G_{S-i}$ respectively, we have that $x_S(C) \geq v^r(C) - \epsilon$ and $x_{S-i}(C) \geq v^r(C) - \epsilon$. Therefore,

$$x(C) = \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C) \geq \left[\frac{p'}{p} + 1 - \frac{p'}{p}\right] \cdot (v^r(C) - \epsilon) = v^{r'}(C) - \epsilon,$$

where the second transition uses $p' \leq p$ and the third transition follows since $v^{r'}(C) = v^r(C)$ (part 1 of Lemma 1). Now let $i \in C$. Once again, we have that

$$\begin{aligned} x(C) &= \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C) = \frac{p'}{p} \cdot x_S(C) + \left[1 - \frac{p'}{p}\right] \cdot x_{S-i}(C-i) \\ &\geq \frac{p'}{p} \cdot (v^r(C) - \epsilon) + \left[1 - \frac{p'}{p}\right] \cdot (v^r(C-i) - \epsilon) = v^{r'}(C) - \epsilon. \end{aligned}$$

The second transition follows since payment to agent $i$ in $\boldsymbol{x_{S-i}}$ is 0. The third transition follows since $\boldsymbol{x_S}$ and $\boldsymbol{x_{S-i}}$ are $\epsilon$-core imputations of $G_S$ and $G_{S-i}$ respectively and since $p' \leq p$. The last transition follows due to part 2 of Lemma 1.

Hence, we proved that $x(S) = v^{r'}(S)$ and $x(C) \geq v^{r'}(C) - \epsilon$ for every coalition $C \subseteq S$ where $C \neq S$. This proves that $\boldsymbol{x}$ is an $\epsilon$-core imputation of $G'_S$. Since $S \subseteq N$ was selected arbitrarily, we have proved that every subgame of $G'$ has non-empty $\epsilon$-core, i.e., $G'$ is $\epsilon$-totally balanced. ■

[5] It is possible to combine all the cases in the proof of Theorem 1 using Footnote 3. However, a case-wise analysis is presented to avoid any confusion.

Since $\epsilon$-total balancedness is preserved when we decrease a single reliability, we can decrease multiple reliabilities one-by-one and repeatedly apply Theorem 1 to show $\epsilon$-total balancedness in the resulting game, so we obtain the following.

**Corollary 1.** *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an $\epsilon$-totally balanced reliability game and $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} \leq \boldsymbol{r}$. Then $G'$ is $\epsilon$-totally balanced.*

Any reliability extension of a game can be obtained by starting from the base game (equivalent to its reliability extension with the reliability vector $\mathbf{1}$) and decreasing reliabilities as required. Hence Corollary 1 implies that reliability extensions preserve $\epsilon$-total balancedness. Conversely if a game $G$ is not $\epsilon$-totally balanced, it has a subgame $G_S$, which is also a reliability extension with reliability 1 for $i \in S$ and 0 otherwise, having empty $\epsilon$-core. This proves the following.

**Corollary 2.** *For any $\epsilon \geq 0$, a game is $\epsilon$-totally balanced if and only if every reliability extension of the game is $\epsilon$-totally balanced.*

Shapley [23] showed that convex games are totally balanced. Thus Corollary 2 implies that every reliability extension of a convex game is totally balanced. This strengthens a theorem by Bachrach et. al. [1] which states that every reliability extension of a convex game has a non-empty core.[6] We strengthen this further and prove that every reliability extension of a convex game is in fact convex.

### 4.2 Approximately Convex Games

For the reliability extension of $\epsilon$-convex games, the results are parallel to those for $\epsilon$-totally balanced games, but require different proof techniques.

**Theorem 2.** *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an $\epsilon$-convex reliability game. Fix $i \in N$ and let $p = r_i > 0$ be the reliability of agent $i$ in $G$. Take $p'$ such that $0 \leq p' \leq p$ and define $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} = (p', \boldsymbol{r_{-i}})$. Then $G'$ is $\epsilon$-convex.*

*Proof Sketch.* Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ be the characteristic functions of $G$ and $G'$ respectively. We want to prove that $G'$ is $\epsilon$-convex, i.e., for every $j \in N$ and for all $S \subseteq T \subseteq N - j$, $v^{\boldsymbol{r'}}(S + j) - v^{\boldsymbol{r'}}(S) \leq v^{\boldsymbol{r'}}(T + j) - v^{\boldsymbol{r'}}(T) + \epsilon$. We know that this is true for $v^{\boldsymbol{r}}$ since $G$ is $\epsilon$-convex. We analyze the marginal contributions of $j$ to $S$ and $T$ in both $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r'}}$ and apply $\epsilon$-convexity of $G$ and Lemma 1 (wherever required) in order to prove $\epsilon$-convexity of $G'$. ■

The proof appears in the full version of the paper. Similarly to totally balanced games, Theorem 2 can be extended to cover general decreases in reliabilities, including those starting from the base game.

**Corollary 3.** *Let $\epsilon \geq 0$. Let $G = (N, v, \boldsymbol{r})$ be an $\epsilon$-convex reliability game and $G' = (N, v, \boldsymbol{r'})$ where $\boldsymbol{r'} \leq \boldsymbol{r}$. Then $G'$ is $\epsilon$-convex.*

**Corollary 4.** *For any $\epsilon \geq 0$, a game is $\epsilon$-convex if and only if every reliability extension of the game is $\epsilon$-convex.*

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[6] The converse part of Theorem 3 in [1] is technically incorrect and only holds when the game is not totally balanced, which is again generalized by our results.

### 4.3 Relation between convexity and total balancedness

Shapley [23] proved that convex games are totally balanced. In the above results, we deal with the notions of $\epsilon$-convexity and $\epsilon$-total balancedness. We now prove a relation between the two concepts for any $\epsilon \geq 0$, extending Shapley's result.

**Theorem 3.** *For any $\epsilon \geq 0$, an $\epsilon/(n-1)$-convex game with $n$ agents is $\epsilon$-totally balanced.*

The proof of this theorem is along the same lines as the proof of Shapley's result (see, e.g., [7]) and appears in the full version of the paper. We also show that the sufficient condition in Theorem 3 cannot be improved by a factor of more than $n-1$. The proof again appears in the full version of the paper.

**Lemma 2.** *For any $\epsilon \geq 0$, $\delta > 0$ and $n \in \mathbb{N}$, there exists a game with $n$ agents which is $\epsilon + \delta$-convex but not $\epsilon$-totally balanced.*

There are several implications of this relation. First, Corollary 4 showed that any reliability extension of an $\epsilon$-convex game is $\epsilon$-convex. Using Theorem 3, we can see that such an extension is also $\epsilon \cdot (n-1)$-totally balanced. Second, the core has been well studied in the literature. For simple games, the core is non-empty if and only if a veto agent exists. In general games, convexity serves as a sufficient condition for non-emptiness of the core. However, conditions for non-emptiness of the approximate core are relatively less studied. Theorem 3 provides such a sufficient condition in terms of approximate convexity.

## 5 Computing a core imputation

In Section 4, we proved that every reliability extension of a totally balanced game is totally balanced and thus has a non-empty core. However, the proof was non-constructive. For several classes of totally balanced games without failures, elegant LP based approaches exist to compute a core imputation in polynomial time. But computing a core imputation in the reliability extension may have a different computational complexity. For example, Bachrach et. al. [1] note that although computing a core imputation is easy in connectivity games on networks, even computing the value of a coalition (and hence computing a core imputation) becomes computationally hard in the reliability extension.

Nevertheless, we show that it is possible to compute a core imputation in any reliability extension of a totally balanced game with high probability. In this section, we use $\epsilon$-core for both $\epsilon \geq 0$ (core/approximate core) and $\epsilon < 0$ (superstable core). In literature, the approximate core is well studied in cases where the core is empty. When the core is not empty, typically only the least core, which corresponds to the $\epsilon_{\min}$-core ($\epsilon_{\min} < 0$) is studied.

First, we show how to compute the core (or the $\epsilon$-core) in a reliability extension of a general game in terms of the core (or the $\epsilon$-core) of the subgames of the base game. The latter is known to be a tractable problem for many domains.

**Theorem 4.** *Let $\epsilon \in \mathbb{R}$. Let $G = (N, v)$ be an $\epsilon$-totally balanced game and $G^{\boldsymbol{r}} = (N, v, \boldsymbol{r})$ be its reliability extension. For any coalition $S \subseteq N$, let $\boldsymbol{x_S}$ be an $\epsilon$-core imputation of the subgame $G_S$. Define $\boldsymbol{x^*} = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S}$, where $\Pr(S|N) = \prod_{i \in S} r_i \cdot \prod_{i \in N \setminus S}(1 - r_i)$. Then the following holds.*

1. *If $\epsilon \geq 0$, then $\boldsymbol{x^*}$ is an $\epsilon$-core imputation of $G^{\boldsymbol{r}}$.*
2. *If $\epsilon < 0$, then $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon$-core imputation of $G^{\boldsymbol{r}}$, where $r_{\min} = \min_{i \in N} r_i$.*

*Proof.* For every coalition $S$, by definition we have that $x_S(S) = v(S)$ and $x_S(C) \geq v(C) - \epsilon$ for every $C \subseteq S$. Let $v^{\boldsymbol{r}}$ be the characteristic function of $G^{\boldsymbol{r}}$. First, we prove that $\boldsymbol{x^*}$ is a pre-imputation of $G^{\boldsymbol{r}}$.

$$x^*(N) = \sum_{S \subseteq N} \Pr(S|N) \cdot x_S(S) = \sum_{S \subseteq N} \Pr(S|N) \cdot v(S) = v^{\boldsymbol{r}}(N),$$

where the first transition follows since payment to agents in $N \setminus S$ is zero in $\boldsymbol{x_S}$, the second transition follows since $x_S(S) = v(S)$ and the last transition follows due to Equation (1). Now, fix any coalition $C \subseteq N$ where $C \neq N$. For any $C' \subseteq C$, for all $S \subseteq N$ such that $S \cap C = C'$, we have $x_S(C) = x_S(C') \geq v(C') - \epsilon$ except when $S = C'$ where we have $x_{C'}(C) = x_{C'}(C') = v(C')$. Now,

$$\begin{aligned}
x^*(C) &= \sum_{S \subseteq N} \Pr(S|N) \cdot x_S(C) = \sum_{C' \subseteq C} \left[ \sum_{S \subseteq N \, s.t. \, S \cap C = C'} \Pr(S|N) \cdot x_S(C) \right] \\
&\geq \sum_{C' \subseteq C} \left[ \left( \sum_{\substack{S \subseteq N \, s.t. \\ S \cap C = C', S \neq C'}} \Pr(S|N) \cdot (v(C') - \epsilon) \right) + \Pr(C'|N) \cdot v(C') \right] \\
&= \sum_{C' \subseteq C} \left[ (v(C') - \epsilon) \cdot \left( \sum_{S \subseteq N \, s.t. \, S \cap C = C'} \Pr(S|N) \right) + \Pr(C'|N) \cdot \epsilon \right] \\
&= \sum_{C' \subseteq C} \left[ (v(C') - \epsilon) \cdot \Pr(C'|C) + \Pr(C'|N) \cdot \epsilon \right] \\
&= \sum_{C' \subseteq C} \left[ \Pr(C'|C) \cdot v(C') \right] - \epsilon \cdot \left[ \sum_{C' \subseteq C} \Pr(C'|C) - \sum_{C' \subseteq C} \Pr(C'|N) \right] \\
&= v^{\boldsymbol{r}}(C) - \epsilon \cdot \left( 1 - \prod_{i \in N \setminus C} (1 - r_i) \right),
\end{aligned}$$

where the fourth transition follows by adding and subtracting $\Pr(C'|N) \cdot \epsilon$ in the outer summation and rearranging terms and the last transition follows from Equation (1). Formal proofs for intuitive substitutions used in the fifth and the last transitions appear in the full version of the paper.

Using this and that $\boldsymbol{x^*}$ is a pre-imputation, we know that $\boldsymbol{x^*}$ is an $\epsilon'$-core imputation of $G^{\boldsymbol{r}}$ if $\epsilon' \geq \epsilon \cdot \left(1 - \prod_{i \in N \setminus C}(1 - r_i)\right)$, for every $C \subseteq N$ such that

$C \neq N$. If $\epsilon \geq 0$, then we need to maximize $1 - \prod_{i \in N \setminus C}(1 - r_i)$ else we need to minimize it. A trivial upper bound is $1 - \prod_{i \in N}(1 - r_i) \leq 1$. We use the loose upper bound of 1. For a lower bound, note that since $C \neq N$, there exists $j \in N \setminus C$, hence $\prod_{i \in N \setminus C}(1 - r_i) \leq 1 - r_j \leq 1 - r_{\min}$. Thus $r_{\min}$ is a lower bound (which is also attained when $C = N - t$ where $r_t = r_{\min}$). This proves that $\boldsymbol{x^*}$ is an $\epsilon$-core imputation of $G^{\boldsymbol{r}}$ if $\epsilon \geq 0$, and an $r_{\min} \cdot \epsilon$-core imputation if $\epsilon < 0$. ■

For a game $G = (N, v)$, define $\epsilon^*$ as the maximum least core value over all subgames of $G$. That is, $\epsilon^*(G) = \max_{S \subseteq N} \epsilon_{\min}(G_S)$. Note that a subgame with a single agent has $\epsilon_{\min} = -\infty$ by definition. Thus, every subgame of $G$ has non-empty $\epsilon^*$-core and hence there exists an $\epsilon^*$-core imputation for every subgame. Since an $\epsilon$-core imputation is also an $\epsilon^*$-core imputation for any $\epsilon \leq \epsilon^*$ (by definition), any least core imputation of any subgame of $G$ is also an $\epsilon^*$-core imputation of that subgame. Thus we obtain the following.

**Corollary 5.** *Let $G$, $G^{\boldsymbol{r}}$ and $r_{\min}$ be as defined in Theorem 4. Let $\epsilon^*$ denote the maximum least core value over all subgames of $G$. For any coalition $S \subseteq N$, let $\boldsymbol{x_S}$ be a least core imputation of $G_S$. Define $\boldsymbol{x^*} = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S}$, where $\Pr(S|N)$ is as defined in Theorem 4. Then we have that*

1. *If $\epsilon^* \geq 0$, then $\boldsymbol{x^*}$ is an $\epsilon^*$-core imputation of $G^{\boldsymbol{r}}$.*
2. *If $\epsilon^* < 0$, then $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$.*

Consider a totally balanced game $G = (N, v)$. Assume that we have an oracle $LC$ such that $LC(G_S)$ returns a least core imputation of the subgame $G_S$ of the base game $G$. Such an oracle subroutine exists with polynomial time complexity for many classes of totally balanced games. For example, Solymosi et. al. [26] give a polytime algorithm to compute the nucleolus of assignment games which are totally balanced [25]. Nucleolus is a special (and unique) least core imputation that maximizes stability. Other examples of polytime algorithms to compute the nucleolus of totally balanced games include the algorithm by Kuipers [16] for convex games, the algorithm by Deng et. al. [9] for simple flow games and the algorithm by Kern et. al. [15] for matching games (which are totally balanced over bipartite graphs [10]). For technical reasons, we extend the payment vector $LC(G_S)$ to a payment vector over all agents by setting the payments to agents in $N \setminus S$ to be zero. For now, we assume that $LC$ is deterministic, in the sense that it returns the same least core imputation every time it is called with the same subgame. We relax this assumption in Remark 2.

Observe that $\epsilon^* \leq 0$ for a totally balanced game. Corollary 5 implies that $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$ (and hence a core imputation as well since $r_{\min} \cdot \epsilon^* \leq 0$) that can be computed using exponentially many calls to $LC$. We reduce the number of calls to $LC$ to a polynomial in $n$, $\log(1/\delta)$ ($1 - \delta$ is the confidence level), $v(N)$, $1/r_{\min}$ and $1/|\epsilon^*|$ by sampling the subgames instead of iterating over them and using some additional tricks. However, both $r_{\min}$ and $|\epsilon^*|$ can be exponentially small (even zero) making this algorithm possibly an exponential time algorithm. Thus if the bound on $k$ in Theorem 5 exceeds $2^n$ or if $\epsilon^* = 0$, we revert to the naïve exponential summation of Corollary 5. Section 7 discusses the issues with computation of $\epsilon^*$. Note that even when $\epsilon^*$ is unknown, the algorithm can be used in practice by taking large number of samples. Also, the algorithm uses the value of $v^{\boldsymbol{r}}(N)$ which is easy to approximate by sampling and the additive error can be taken care of as in Footnote 7.

**Algorithm CoreReliability:** Computing a core imputation of a reliability extension of a totally balanced game.
**Input:** Totally balanced game $G = (N, v)$, subroutine $LC$ to compute a least core imputation of subgames of $G$, reliability vector $\boldsymbol{r}$, $v^{\boldsymbol{r}}(N)$, $\delta$ and $k$.
**Output:** $\hat{\boldsymbol{x}}$, which is in the core of $G^{\boldsymbol{r}}$ with probability at least $1 - \delta$.

1. Set $\boldsymbol{y} = \boldsymbol{0}$.
2. For $t = 1$ to $k$ do
   (a) For each agent $i \in N$, set $l_i = 1$ with probability $r_i$ and $l_i = 0$ otherwise.
   (b) $\boldsymbol{y} = \boldsymbol{y} + LC(G_S)$ where $S = \{i \in N \mid l_i = 1\}$ (the survivor set).
3. Let $\boldsymbol{x} = \boldsymbol{y}/k$.
4. Return $\hat{\boldsymbol{x}} = \boldsymbol{x} - \gamma \cdot \boldsymbol{1}$, where $\boldsymbol{1} = \langle 1, 1, \ldots, 1\rangle$ and $\gamma = \frac{1}{n} \cdot (x(N) - v^{\boldsymbol{r}}(N))$.

**Theorem 5.** *The payment vector $\hat{\boldsymbol{x}}$ returned by Algorithm* CoreReliability *is in the core of $G^{\boldsymbol{r}}$ with probability at least $1 - \delta$ if*

$$k \geq \frac{2 \cdot v(N)^2 \cdot n^2 \cdot \log\left(\frac{2 \cdot n}{\delta}\right)}{r_{\min}^2 \cdot |\epsilon^*|^2}.$$

*Proof.* Let $\boldsymbol{x_S} = LC(G_S)$. In Step 2, every $S$ is sampled with probability $\Pr(S|N)$ and the value added is $\boldsymbol{x_S}$, so $E[\boldsymbol{x}] = \sum_{S \subseteq N} \Pr(S|N) \cdot \boldsymbol{x_S} = \boldsymbol{x^*}$ (as in Corollary 5). For any $S \subseteq N$ and for any $i \in N$, the payment to agent $i$ in $LC(G_S)$ is in $[0, v(N)]$. Using Hoeffding's inequality, for any $i \in N$,

$$\Pr\left(|x_i - x_i^*| \geq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}\right) \leq 2 \cdot e^{-\frac{2 \cdot k}{v(N)^2} \cdot \left(\frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}\right)^2}.$$

Substituting the value of $k$, we get that this probability is at most $\delta/n$ for every $i \in N$. Taking union bound over $i \in N$, we obtain that the probability that

$$\forall\, i \in N, \quad |x_i - x_i^*| \leq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}, \tag{2}$$

holds is at least $1 - \delta$. Now we prove that $\hat{\boldsymbol{x}}$ is a core imputation of $G^{\boldsymbol{r}}$ assuming Equation (2) holds. First of all, we can see that $\sum_{i \in N} x_i \leq \sum_{i \in N} x_i^* + r_{\min} \cdot |\epsilon^*|/2$. But using Corollary 5, we know that $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$ and hence $\sum_{i \in N} x_i^* = v^{\boldsymbol{r}}(N)$. Therefore $\gamma$ in Step 4 of the algorithm follows $\gamma = \frac{1}{n} \cdot \left(\sum_{i \in N} x_i - v^{\boldsymbol{r}}(N)\right) \leq \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n}$. Hence, we can see that for every $i \in N$,

$$\hat{x}_i = x_i - \gamma \geq x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{2 \cdot n} - \gamma \geq x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{n}, \tag{3}$$

where the second transition follows due to Equation (2). Hence for any $C \subseteq N$,

$$\hat{x}(C) = \sum_{i \in C} \hat{x}_i \geq \sum_{i \in C} \left(x_i^* - \frac{r_{\min} \cdot |\epsilon^*|}{n}\right) \geq x^*(C) - r_{\min} \cdot |\epsilon^*|. \tag{4}$$

Since $\boldsymbol{x^*}$ is an $r_{\min} \cdot \epsilon^*$-core imputation of $G^{\boldsymbol{r}}$, we know that $x^*(C) \geq v^{\boldsymbol{r}}(C) - r_{\min} \cdot \epsilon^* = v^{\boldsymbol{r}}(C) + r_{\min} \cdot |\epsilon^*|$ (since $\epsilon^* < 0$). Substituting this into Equation (4), we get that $\hat{x}(C) \geq v^{\boldsymbol{r}}(C)$ for every $C \subseteq N$ and $C \neq N$. Furthermore, $\hat{x}(N) = x(N) - n \cdot \gamma = v^{\boldsymbol{r}}(N)$ by definition of $\hat{\boldsymbol{x}}$ and $\gamma$. Hence $\hat{\boldsymbol{x}}$ is in the core of $G^{\boldsymbol{r}}$. ■

*Remark 1.* Note that the algorithm works so long as the subroutine $LC$ can compute an $\epsilon^*$-core imputation of every subgame of the base game. The reason why we have chosen to work with the least core is that in our case $\epsilon^* < 0$ and when it is possible to compute an $\epsilon^*$-core imputation of every subgame, it is usually possible to compute a least core imputation of every subgame as well.[7]

*Remark 2.* Initially we assumed that $LC$ returns a fixed least core imputation for every subgame. This is because Theorem 4 only works with fixed imputations. However, it is easy to check that if $LC$ has any distribution over the set of all least core imputations of a subgame, then the expected payment vector returned by $LC$ is a least core imputation of that subgame. Thus in Algorithm CoreReliability, $E[\boldsymbol{x}] = E[\boldsymbol{x^*}]$ is still an $r_{\min} \cdot \epsilon^*$-core imputation of the reliability extension, and the algorithm still works with the same bound on $k$.[8]

*Remark 3.* Note that Hoeffding's inequality is usually applied when the requirements for the result are somewhat fuzzy whereas the requirements of a core imputation are quite strict. The use of least core imputations of subgames of the base game provides us enough margin of error to be able to use Hoeffding's inequality and still satisfy the strict constraints with high probability.

## 6 Failures and Non-emptiness of the Core

While studying reliability extensions of totally balanced games, we saw that introducing failures in a totally balanced game without failures, and increasing failure probabilities in a totally balanced reliability game preserve total balancedness, and hence non-emptiness of the core. In this section, we outline the effect of these two operations on three classes of games: i) general cooperative games, ii) totally balanced games, and iii) simple games.

**General Games:** In many games introducing failures does not preserve non-emptiness of the core. Any game $G$ that has a non-empty core but is not totally balanced is such a game since a subgame of $G$ with an empty core is also a reliability extension of $G$. Introducing failures is a special case of increasing failure probabilities where we start with failure probabilities being zero, so increasing failure probabilities also does not preserve core non-emptiness in general games.

**Totally Balanced Games:** Our results show that for totally balanced games, both introducing failures and more generally increasing failure probabilities preserve non-emptiness of the core (in fact, they preserve total balancedness).

[7] $LC$ can also be replaced by a subroutine $LC'$ which returns an approximate least core imputation so long as the additive error in each component is less than $r_{\min} \cdot \epsilon^*/n$.

[8] Algorithm CoreReliability can be easily adapted to compute an approximate or superstable core imputation of any reliability game in general with high probability.

**Simple Games:** For simple games, Bachrach et. al. [1] observe that introducing failures preserves non-emptiness of the core. To analyze increasing failure probabilities, we performed simulations on a special class of simple games known as weighted voting games. A weighted voting game is defined by $G = (N, \boldsymbol{w}, t)$ where $N$ is a set of agents where each agent $i \in N$ has a weight $w_i \geq 0$, $\boldsymbol{w}$ is the vector of these weights and $t$ is the threshold; a coalition $C$ with $\sum_{i \in C} w_i \geq t$ has $v(C) = 1$ and $v(C) = 0$ otherwise. Simulations revealed the following example where increasing failure probabilities does not preserve core non-emptiness.

**Example:** Consider a weighted voting game $G$ with 5 agents with weight vector $\boldsymbol{w} = \langle 4, 3, 3, 2, 1 \rangle$ and threshold $t = 6$. Consider its reliability extension $G^r$ with the reliability vector $\boldsymbol{r} = \langle 0.1, 0.6, 1, 1, 0.5 \rangle$. $G^r$ has a non-empty core, but decreasing the reliability of agent 5 from 0.5 to 0.1 makes the core empty.

While the theme that decreasing reliability increases stability does not hold strictly for simple games, it appears to hold on average, at least for weighted voting games. We performed several simulations where we randomly generated weighted voting games with weights sampled from various distributions, e.g., uniform distribution, normal distribution, exponential distribution etc. We kept the reliabilities of all the agents equal, and observed that as this uniform reliability decreases (i.e., the failure probability increases), the probability of the core being non-empty increases. We also observed the same result for the $\epsilon$-core.

## 7 Discussion and Future Work

We studied the reliability extension of totally balanced games. We proved that both $\epsilon$-total balancedness and $\epsilon$-convexity (generalizations of the respective concepts) are preserved when the reliabilities decrease. We proved a relation between these classes, generalizing a result by Shapley [23] that ties convexity and total balancedness. We also proposed an algorithm to compute a core imputaiton of any reliability extension of a totally balanced game with high probability.

This opens several possibilities for future research. First, Lemma 1 shows how the reliabilities affect the characteristic function of a game and we derived some useful results about totally balanced games and convex games building on it. Lemma 1 might also have other applications, e.g., in analyzing the effect of failures on power indices such as the Shapley value or the Banzhaf power index. It would also be interesting to examine how the reliability extension affects the external subsidy required to maintain stability, i.e. the Cost of Stability [3,21].

Next, the number of samples required in the algorithm presented in Section 5 depends on $\epsilon^*$, the maximum least core value over all subgames of the game without failures. We are unable to settle the question of computing $\epsilon^*$ or obtaining a lower bound on it in polynomial time (and thus obtaining an upper bound on the number of samples required). Such an investigation may also lead to discoveries regarding the relative stabilities of different subgames of a cooperative game.

Lastly, our analysis is restircted to games where only one coalition can be formed. In contrast, cooperative games with coalitional structures [17] allow multiple coalitions to arise simultaneously, and are used successfully to model collaboration in multi-agent environments [22,6,2]. It would be interesting to extend the reliability extension model to games with coalitional structures.

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# Appendix

## A Proof of Lemma 1

*Proof.* For the first part, notice that if $i \notin S$ then in Equation (1), $i \notin S'$ for every $S' \subseteq S$. Hence $v^{\boldsymbol{r}}(S)$ is independent of $r_i$ and does not change when $r_i$ is changed from $p$ to $p'$. For the second part, observe that

$$\begin{aligned} v^{\boldsymbol{r}}(S) &= \sum_{S' \subseteq S} \Pr(S'|S) \cdot v(S') = \sum_{S' \subseteq S} \left( \prod_{j \in S'} r_j \cdot \prod_{j \in S \setminus S'} (1 - r_j) \right) \cdot v(S') \\ &= \sum_{S' \subseteq S | i \in S'} \left( r_i \cdot \prod_{j \in S' - i} r_j \cdot \prod_{j \in S \setminus S'} (1 - r_j) \right) \cdot v(S') \\ &\quad + \sum_{S' \subseteq S - i} \left( (1 - r_i) \cdot \prod_{j \in S'} r_j \cdot \prod_{j \in (S-i) \setminus S'} (1 - r_j) \right) \cdot v(S'). \end{aligned} \tag{5}$$

The last transition follows by breaking the summation into two parts: subsets containing $i$ and subsets not containing $i$. Now for any coalition $S \subseteq N$ such that $i \in S$, define $v_i^{rel}(S)$ as the value of $S$ in the game $G_i^{rel} = (N, v, \boldsymbol{r_i^{rel}})$ where $\boldsymbol{r_i^{rel}} = (1, \boldsymbol{r_{-i}})$. Hence we replace $r_i$ by 1 in Equation (5) to obtain $v_i^{rel}(S)$. Thus,

$$v_i^{rel}(S) = \sum_{S' \subseteq S | i \in S'} \left( \prod_{j \in S' - i} r_j \cdot \prod_{j \in S \setminus S'} (1 - r_j) \right) \cdot v(S'). \tag{6}$$

Moreover, using Equation (1), we have that

$$v^{\boldsymbol{r}}(S - i) = \sum_{S' \subseteq S - i} \left( \prod_{j \in S'} r_j \cdot \prod_{j \in (S-i) \setminus S'} (1 - r_j) \right) \cdot v(S'). \tag{7}$$

Substituting Equations (6) and (7) in Equation (5) and observing that $r_i$ = p, we get $v^{\boldsymbol{r}}(S) = p \cdot v_i^{rel}(S) + (1-p) \cdot v^{\boldsymbol{r}}(S - i)$. Similarly, $v^{\boldsymbol{r}}(S) = p' \cdot v_i^{rel}(S) + (1 - p') \cdot v^{\boldsymbol{r}'}(S - i)$. Observing that $v^{\boldsymbol{r}}(S - i) = v^{\boldsymbol{r}'}(S - i)$ (using the first part) and eliminating $v_i^{rel}(S)$ from the two equations, we get the desired result. ■

## B Proof of Theorem 2

*Proof.* Let $v^{\boldsymbol{r}}$ and $v^{\boldsymbol{r}'}$ denote the characteristic functions of $G$ and $G'$ respectively. We want to prove that $G'$ is $\epsilon$-convex. For this, we need to show that for any agent $j \in N$ and for all coalitions $S$ and $T$ such that $S \subseteq T \subseteq N - j$, we have $v^{\boldsymbol{r}'}(S + j) - v^{\boldsymbol{r}'}(S) \leq v^{\boldsymbol{r}'}(T + j) - v^{\boldsymbol{r}'}(T) + \epsilon$. We take two cases: $j = i$ and $j \neq i$.

**Case 1. $j = i$ :** Using Lemma 1 we see that

$$\begin{aligned} v^{r'}(S+i) - v^{r'}(S) &= \frac{p'}{p} \cdot v^r(S+i) + \left[1 - \frac{p'}{p}\right] \cdot v^r(S) - v^r(S) \\ &= \frac{p'}{p} \cdot (v^r(S+i) - v^r(S)), \end{aligned}$$

and similarly,

$$v^{r'}(T+i) - v^{r'}(T) = \frac{p'}{p} \cdot (v^r(T+i) - v^r(T)).$$

Now using $\epsilon$-convexity of $G$, we have that $v^r(S+i) - v^r(S) \le v^r(T+i) - v^r(T) + \epsilon$ and thus $v^{r'}(S+i) - v^{r'}(S) \le v^{r'}(T+i) - v^{r'}(T) + {p'}/{p} \cdot \epsilon \le v^{r'}(T+i) - v^{r'}(T) + \epsilon$ (since $p'/p \le 1$ and $\epsilon \ge 0$), as required.

**Case 2. $j \ne i$:** When $j \ne i$, we take three subcases. ($i \notin S$ and $i \notin T$), ($i \notin S$ and $i \in T$), and finally, ($i \in S$ and $i \in T$). Note that the case ($i \in S$ and $i \notin T$) is not feasible since $S \subseteq T$.

**Case 2.1. $i \notin S$, $i \notin T$ :** In this case, since $v^{r'}$ and $v^r$ are equal for all four sets under consideration ($S, S+j, T$ and $T+j$), $\epsilon$-convexity of $G$ implies the desired result.

**Case 2.2. $i \notin S$, $i \in T$ :** In this case, note that $S \subseteq T - i \subseteq T$. Now again using Lemma 1,

$$\begin{aligned} & v^{r'}(T+j) - v^{r'}(T) \\ &= \frac{p'}{p} \cdot (v^r(T+j) - v^r(T)) + \left[1 - \frac{p'}{p}\right] \cdot (v^r(T+j-i) - v^r(T-i)) \\ &\ge \frac{p'}{p} \cdot (v^r(S+j) - v^r(S) - \epsilon) + \left[1 - \frac{p'}{p}\right] \cdot (v^r(S+j) - v^r(S) - \epsilon) \\ &= v^r(S+j) - v^r(S) - \epsilon = v^{r'}(S+j) - v^{r'}(S) - \epsilon, \end{aligned}$$

where the first and the last transitions follow from Lemma 1 and the second transition follows since $G$ is $\epsilon$-convex and since $1 - p'/p \ge 0$ (as $p' \le p$).

**Case 2.3. $i \in S$, $i \in T$ :** As in Case 2.2, we have that

$$v^{r'}(T+j) - v^{r'}(T) = \frac{p'}{p} \cdot (v^r(T+j) - v^r(T)) + \left[1 - \frac{p'}{p}\right] \cdot (v^r(T-i+j) - v^r(T-i))$$

and

$$v^{r'}(S+j) - v^{r'}(S) = \frac{p'}{p} \cdot (v^r(S+j) - v^r(S)) + \left[1 - \frac{p'}{p}\right] \cdot (v^r(S-i+j) - v^r(S-i)).$$

Now, using $\epsilon$-convexity of $G$, we know that $v^r(S+j) - v^r(S) \le v^r(T+j) - v^r(T) + \epsilon$ and $v^r(S-i+j) - v^r(S-i) \le v^r(T-i+j) - v^r(T-i) + \epsilon$. Substituting that, we get that $v^{r'}(S+j) - v^{r'}(S) \le v^{r'}(T+j) - v^{r'}(T) + \epsilon$ as required.

Thus, $G'$ is $\epsilon$-convex, as required. ■

*Remark 4.* Technically, using the generic version of Lemma 1 (see Footnote 3), it is possible to extend the argument for Case 2.3 in the proof of Theorem 2 to cover all the cases. A case-by-case analysis is nevertheless presented to avoid any confusion.

## C Proof of Theorem 3 and Lemma 2

### C.1 Proof of Theorem 3

*Proof.* Fix any $\epsilon \geq 0$. Let $G = (N, v)$ be an $\epsilon/(n-1)$-convex game where $N = \{1, 2, \ldots, n\}$. Consider the marginal contribution vector $\boldsymbol{x}$ defined by $x_1 = v(\{1\})$ and $x_i = v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$ for $i = 2, 3, \ldots, n$. We would like to show that $\boldsymbol{x}$ is an $\epsilon$-core imputation of $G$. First, it is easy to check that $x(N) = \sum_{i=1}^{n} x_i = v(N)$. Now take any coalition $C = \{i_1, i_2, \ldots, i_k\} \subseteq N$ where $C \neq N$ (thus $k \leq n-1$) and assume $i_1 < i_2 < \ldots < i_k$ without loss of generality. We want to show that $x(C) \geq v(C) - \epsilon$. First, note that

$$v(C) = v(\{i_1\}) + \sum_{l=2}^{k} \left(v(\{i_1, \ldots, i_l\}) - v(\{i_1, \ldots, i_{l-1}\})\right). \tag{8}$$

Since $G$ is $\epsilon/(n-1)$-convex, we have

$$v(\{i_1\}) \leq v(\{1, 2, \ldots, i_1 - 1, i_1\}) - v(\{1, 2, \ldots, i_1 - 1\}) + \frac{\epsilon}{n-1} = x_{i_1} + \frac{\epsilon}{n-1},$$

when $i_1 > 1$. Also for $i_1 = 1$, $v(\{1\}) = x_1 \leq x_1 + \epsilon/(n-1)$. Similarly for any $l \in \{2, \ldots, k\}$, we have

$$\begin{aligned} &v(\{i_1, \ldots, i_{l-1}, i_l\}) - v(\{i_1, \ldots, i_{l-1}\}) \\ &\quad \leq v(\{1, 2, \ldots, i_l - 1, i_l\}) - v(\{1, 2, \ldots, i_l - 1\}) + \frac{\epsilon}{n-1} = x_{i_l} + \frac{\epsilon}{n-1}. \end{aligned}$$

Substituting these in Equation (8), we see that $v(C) \leq p(C) + k \cdot \epsilon/(n-1) \leq p(C) + \epsilon$ as $k \leq n-1$ and $\epsilon \geq 0$. ∎

### C.2 Proof of Lemma 2

*Proof.* Let $\epsilon \geq 0$, $\delta > 0$ and $n \in \mathbb{N}$ be given. Consider a game $G = (N, v)$ where $N = \{1, 2, \ldots, n\}$ and $v : 2^N \to \mathbb{R}$ is such that $v(N) = 1$, $v(N-1) = 1 + \epsilon + \delta$ and $v(S) = 0$ for $S \subseteq N$ such that $S \neq N$ and $S \neq N - 1$. We show that $G$ is $\epsilon + \delta$-convex but not $\epsilon$-totally balanced.

To see that $G$ is $\epsilon + \delta$-convex, we need to prove that $v(S + i) - v(S) \leq v(T + i) - v(T) + \epsilon + \delta$ for every agent $i \in N$ and for all $S \subseteq T \subseteq N - i$. For $i = 1$, this holds because the only case when $v(S + 1) - v(S) \leq v(T + 1) - v(T)$ gets violated is when $T = N - 1$ and $S \subseteq T$ but $S \neq T$, in which case we still have $v(S+i) - v(S) \leq v(T+i) - v(T) + \epsilon + \delta$. For any agent $j \in N - 1$, the only case when $v(S+j) - v(S) \leq v(T+j) - v(T)$ gets violated is when $S = N - 1 - j$

and $T = N - j$, in which case we still have $v(S+i) - v(S) \leq v(T+i) - v(T) + \epsilon + \delta$. Therefore, $G$ is $\epsilon + \delta$-convex.

Now to see why $G$ is not $\epsilon$-totally balanced, we show that the $\epsilon$-core of the game itself is empty. Note that for any imputation $\boldsymbol{p}$ to be in the $\epsilon$-core, we require $p(N) = v(N) = 1$ and $p(N-1) \geq v(N-1) - \epsilon = 1 + \delta$. This is not possible since $\delta > 0$. ■

## D Observations used in the proof of Theorem 4

Let $\boldsymbol{r}$ be a reliability vector and for any coalitions $C$ and $S$ such that $C \subseteq S$, let $\Pr(C|S) = \prod_{j \in C} r_j \cdot \prod_{j \in S \setminus C} (1 - r_j)$ be the probability of $C$ being the survivor set of the coalition $S$.

**Observation 6** *For any coalition $S \subseteq N$, we have $\sum_{C \subseteq S} \Pr(C|S) = 1$.*

Consider the set of events $\{E_C | C \subseteq S\}$ where $E_C$ denotes the event that $C$ is the survivor set of $S$. This is a partition of the whole sample space and thus Observation 6 follows from the law of total probability.

**Observation 7** *For any coalition $C \subseteq N$, we have*

$$\sum_{C' \subseteq C} \Pr(C'|N) = \prod_{i \in N \setminus C} (1 - r_i).$$

Similarly to Observation 6, consider the event $A$ denoting that every agent in $N \setminus C$ fails. First of all, the right hand side in Observation 7 is the probability of the event $A$ because the failures are independent. Now consider the set of events $\{A_{C'} | C' \subseteq C\}$ where $A_{C'}$ denotes the event that $C'$ is the survivor set of $N$. It is easy to check that this is a partition of the event $A$ because every agent in $N \setminus C$ fails if and only if the survivor set of $N$ is a subset of $C$. Again, Observation 7 follows from the law of total probability.

**Observation 8** *For any coalitions $C$ and $C'$ such that $C' \subseteq C$, we have*

$$\sum_{S \subseteq N \text{ s.t. } S \cap C = C'} \Pr(S|N) = \Pr(C'|C).$$

To see why Observation 8 holds, consider the event $B$ denoting that every agent in $C'$ survives and every agent in $C \setminus C'$ fails. Consider the set of events $\{B_S | S \subseteq N, S \cap C = C'\}$ where $B_S$ denotes the event that $S$ is the survivor set of $N$. It is again easy to check that this is a partition of the event $B$ because every agent in $C'$ survives and every agent in $C \setminus C'$ fails if and only if $S$ (the survivor set of $N$) contains all agents from $C'$ and none from $C \setminus C'$, i.e., $S \cap C = C'$. Again, Observation 8 follows from the law of total probability.