CSC336 Exam Review Fall 2002

1. An approximation to $f'(x)$ could be determined by calculating $\frac{f(x+h)-f(x)}{h}$ for small values of h.

   Suppose we were to use floating-point arithmetic to calculate this approximation to $f'(x)$. State two factors that we should be aware of that would limit how small a value of h we could use in practice.

2. Let x be a binary double-precision number. Approximately how large can x be before the calculation of $\exp(x)$ overflows? The numbers in this binary double-precision arithmetic can be expressed as $\pm 0.b_1 b_2 \ldots b_{53} \times 2^e$, where $b_1 = 1$ and $b_i = 0, 1$, for $i = 2, 3, \ldots, 53$ and $-1021 \le e \le 1024$.

3. Consider solving the linear system $Ax = b$ on a computer. If b and A are known to be exact but $\|r\|/\|b\| = 10^{-10}$, where r is the residual, how large of a condition number, $\kappa(A)$, can we tolerate if we require a computed solution with relative error at most 10^{-5}?

4. The polynomial $p(x) = 2 - (x+1) + x(x+1)$ interpolates the first three points in the table:

x	-1	0	1	2
y	2	1	2	-7

   By adding one additional term to $p(x)$, find a polynomial that interpolates the whole table.

5. Consider the function

   $$f(x) = \begin{cases} 2(x+1) + (x+1)^3 & x \in [-1, 0], \\ 3 + 5x + 3x^2, & x \in [0, 1], \\ 11 + 11(x-1) + 3(x-1)^2 - (x-1)^3, & x \in [1, 2]. \end{cases}$$

   and suppose that you need to determine a very accurate approximation to

   $$I(f) = \int_{-1}^{2} f(x)dx.$$

   Explain in detail the algorithm that you would use to determine an approximation to $I(f)$. What quadrature rule would you propose using? Why? (Assume that you do not know how to integrate this analytically!)

6. (a) When fitting a large number of data points, what is the main advantage of piecewise polynomial interpolation over interpolation by a single polynomial?

   (b) You need to solve a mathematical problem that has a condition number of 1. Do you consider this problem to be well-conditioned or ill-conditioned? Why?

(c) Under what circumstances does a small residual vector r imply that x is an accurate solution to the linear system $Ax = b$?

7. Suppose we interpolate the function $f(x) = e^{x-1}$ with a polynomial $p(x)$ of degree 9 using 10 interpolation points in $[-1, 1]$.

   (a) Derive an upper bound for the approximation error $|f(x) - p(x)|$ on $[-1, 1]$.

   (b) Suppose the interpolating points were taken to be roots of the Chebychev polynomial $T_{10}(x)$. Derive a new, tighter, upper bound on the approximation error.

8. There are two Newton-Cotes quadrature rules based on $n = 2$ points for integrals over the interval $[-1, 1]$.

   (a) Write down the two rules.

   (b) Describe a problem for which you think that your first rule would be preferable to your second rule.

   (c) Describe a problem for which you think that your second rule would be preferable to your first rule.

9. Find the natural cubic spline function $S(x)$ having the knots -1,0, and 1 that takes the values $S(-1) = 13, S(0) = 7$ and $S(1) = 9$.

10. Consider the problem of approximating the function $f(x) = e^x$ on the interval $x \in [-0.3, 0]$. Suppose we want to find an approximation $S(x)$ that interpolates the data set $\{(-0.3, f(-0.3)), (-0.2, f(-0.2)), (-0.1, f(-0.1)), (-0.0, f(0.0))\}$. If you require $S(x)$ such that

$$|f(x) - S(x)| < 1.0 \times 10^{-3},$$

would you recommend approximating $f(x)$ with a cubic polynomial or a cubic spline?

11. Define the functions $s_1(x)$ and $s_2(x)$ as follows:

$$s_1(x) = x^3, \quad s_2(x) = a + b(x-1) + c(x-1)^2 + \frac{1}{2}(x-1)^3.$$

And let

$$S(x) = \begin{cases} s_1(x), & x \in [0, 1], \\ s_2(x), & x \in [1, 3]. \end{cases}$$

Determine the value of (a, b, c) that makes $S(x)$ a cubic spline. Is $S(x)$ a natural spline?

12. Consider the definite integral

$$I = \int_{2.80}^{2.85} \ln(x) dx.$$

(a) Estimate the value of I by using the trapezoidal rule and rounded, five decimal-digit arithmetic.

(b) You can evaluate this integral by using the following formula

$$\int \ln(x)dx = x\ln(x) - x + C,$$

where C is a constant. Use this formula and rounded, five decimal-digit arithmetic, to compute an estimate to the value of I.

(c) Your answer in part(a) comes from evaluating an approximation formula while your answer in part (b) comes from evaluating an exact formula for the definite integral. Which result would you trust more?