UNIVERSITY OF TORONTO
Faculty of Arts and Science

AUGUST EXAMINATIONS 2002

CSC 336H1 Y

Duration - 2 hours

Examination Aids: Two 8.5×11 handwritten cheat sheets,
A non-programmable calculator.

1. [10]

   (a) Consider the linear system $Ax = b$, where A is an $n \times n$ non-singular symmetric matrix and x, b are $n \times 1$ vectors. Describe how to efficiently solve for x. What is the number of flops used?

   (b) Consider a normalized base-2 3-bit mantissa floating point system. If the exponent is bounded by 4, what is the largest floating point number? Write the number in base-10 form.

   (c) List 2 important characteristics of a degree k B-spline.

   (d) What is an advantage of using Lagrange polynomial representation over monomial polynomial representation? What is an advantage of using monomial polynomial representation over Lagrange polynomial representation?

   (e) Show that $f(x) = x$ and $g(x) = 2x^2 - 1$ are orthogonal in $[-1, 1]$ with respect to weight function $w(x) = 1/\sqrt{1 - x^2}$.

2. [10] Assume that x and y are normalized positive floating point numbers in a base-2 computer with t-bit mantissa. How small can $y - x$ be if $x < 8 < y$?

3. [7] Consider the $2n \times 2n$ system

$$\begin{bmatrix} A & Z \\ A & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

   where A, B are $n \times n$ non-singular matrices, Z is an $n \times n$ matrix with entries all zero, b_1, b_2 are $n \times 1$ vectors, and x_1, x_2 are $n \times 1$ unknown vectors. Describe how to efficiently solve for x_1 and x_2. What is the number of flops used?

4. (a) [4] Let $p_3(x)$ be the polynomial interpolant of $f(x) = x^5$ on the points $\{(-2, f(-2)), (-1, f(-1)), (1, f(1)), (2, f(2))\}$. Find $p_3(x)$ using any method discussed in class.

(b) [6] Find the best bound that you can on the approximation error

$$\max_{-2\le x\le 2} |f(x) - p_3(x)|.$$

(c) [3] Suppose that the interpolating points were taken to be the roots of the Chebychev polynomial $T_4(x)$ for the interval $[-2, 2]$. Derive a new, tighter, upper bound on the approximation error.

5. [10] Let $\{l_i(x)\}_{i=0}^{n}$ be the Lagrange basis functions for the set of nodes $\{x_i\}_{i=0}^{n}$. Show that

$$\sum_{i=0}^{n} l_i(x) = 1.$$

6. [10] Let $S(x)$ be the **not-a-knot** cubic spline interpolant of $\{(0, -2), (1, 0), (2, 10), (3, 34)\}$. Find $S(x)$. What is $S(1.5)$?

7. (a) [8] Determine α, β and γ, such that the quadrature rule formula of the type

$$\int_0^1 x^a f(x)dx \approx \alpha f(0) + \beta f'(0) + \gamma f(1)$$

has maximum polynomial degree. The constants α, β and γ are to be given in terms of a, a given constant such that $a > -1$.

(b) [3] What is the polynomial degree of the formula based on the constants obtained in (a)?

(c) [3] Transform the formula in (a) into one that is appropriate for approximating $\int_0^h t^a g(t)dt$.

8. The Compound Simpson rule with n panels for approximating $I = \int_a^b f(x)dx$ is given by $\frac{h}{6}[f(a) + 2\sum_{i=1}^{n-1} f(a + ih) + 4\sum_{i=1}^{n} f(a + (i - \frac{1}{2})h) + f(b)]$ with error $-f^{(4)}(\eta)\frac{h^4}{2880}(b - a)$.

(a) [2] Let S_n be an approximation of I by the Compound Simpson's rule with n panels and $E_n = I - S_n$ be its error. Derive an estimation of E_{2n} based on S_n and S_{2n}.

(b) [4] Consider the integral

$$I = \int_0^{\pi/2} \sin(x)dx.$$

Use the Compound Simpson's rule with 2 panels to approximate I. Use part (a) to estimate the error of the approximation.

Total Pages = 2

Total Marks = 80

Have a good summer!