Polynomial Interpolation at Chebychev pts

Chebychev polynomials in $[-1,1]$.

• $T_0(x) = 1, T_1(x) = x, T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$, $k \geq 1$ ($T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, \ldots$)

• $T_k(x)$ k-th degree

• $T_k(x) = \cos(k \cos^{-1}(x))$

• Leading coef of $T_k = 2^{k-1}$

• Roots $t_i = \cos(\frac{(2i+1)\pi}{2k})$, $0 \leq i \leq k-1$
These **Chebychev pts** are: distinct; do not include $-1, 1$; concentrated towards end pts $-1, 1$

• $\max_{-1 \leq x \leq 1} |T_k(x)| = 1$

Theorem:
Let $w(x) = (x - x_0)(x - x_1)\ldots(x - x_n)$. Among all possible choices for distinct $\{x_j\}_{j=0}^n$ in $[-1,1]$, $\max_{-1 \leq x \leq 1} |w(x)|$ is minimized if $w(x) = \frac{1}{2^n} T_{n+1}(x)$, i.e. if x_j are the roots of $T_{n+1}(x)$
($x_j = t_j, \min_{x_i} \max_{-1 \leq x \leq 1} |w(x)| = \frac{1}{2^n}$).

Therefore, if we can choose the interpolation pts in $[-1,1]$ the best choice as far as minimizing the max. interp. error in $[-1,1]$ is the Chebychev pts t_i.

For a general interval $[a,b]$ translate $[a,b]$ into $[-1,1]$.
•$t \in [-1,1], u \in [a,b], u = \frac{b-a}{2}t + \frac{a+b}{2}$
or $t = \frac{2(u-a)}{b-a} - 1$
•$t_i \in [-1,1], u_i = \frac{b-a}{2}t_i + \frac{a+b}{2} \in [a,b]$
•$\hat{T}_k(u) = T_k(t) = T_k(\frac{2(u-a)}{b-a} - 1)$
Leading coef of $\hat{T}_k(u) = 2^{k-1}(\frac{2}{b-a})^k$
• $\min_{x_i} \max_{a \leq x \leq b} |w(x)| = \frac{1}{2^n}(\frac{b-a}{2})^{n+1}$

Let $t_i = \cos(\frac{(2i+1)\pi}{2n+2})$ and $x_i = -1 + \frac{2}{n} \cdot i, i = 0, \ldots, n$
Let $q_n(x)$ and $p_n(x)$ be the interpolating polynomials of degree $\leq n$ constructed based on data $(t_i, g(t_i))$ and $(x_i, g(x_i)), i = 0, \ldots, n$, respectively.

The Theorem guarantees that

$$\max_{-1 \leq x \leq 1} |(x - t_0)(x - t_1)\ldots(x - t_n)| \\ \leq \max_{-1 \leq x \leq 1} |(x - x_0)(x - x_1)\ldots(x - x_n)|$$

It does **not** guarantee that for any x in $[-1,1]$

$$|(x - t_0)(x - t_1)\ldots(x - t_n)| \\ \leq |(x - x_0)(x - x_1)\ldots(x - x_n)|$$

Moreover, it does **not** guarantee that

$$\max_{-1 \leq x \leq 1} |g(x) - q_n(x)| \leq \max_{-1 \leq x \leq 1} |g(x) - p_n(x)|$$

However, in practice this is very likely.

Numerical Integration

It is often required to calculate the value of $I = \int_a^b f(x)dx$. If f is given in a complicated analytical form, or if f is given only in tabular form, the standard techniques for integration may not be applicable. In this case we apply numerical techniques to compute an approximation Q to I.

A method for computing an approx Q to I is called a **quadrature rule**. Many quadrature rules are developed by first approximating $f(x)$ by a simpler function $p(x)$, say a polynomial, then taking Q as approx to I,

$$Q = \int_a^b p(x)dx \approx \int_a^b f(x)dx = I$$

When $p(x)$ is an interpolating poly of f, the quadrature rule is called **interpolatory**.

Assume we have data $\{(x_i, f(x_i)), i = 0, \ldots, n\}$, with x_i distinct. Then $p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$ is the poly of degree $\leq n$, interpolating the above data.

The respective quadrature rule is

$$\begin{aligned} I &= \int_a^b f(x)dx \approx \int_a^b p_n(x)dx \\ &= \int_a^b \sum_{i=0}^{n} f(x_i) l_i(x)dx \\ &= \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)dx \\ &= \sum_{i=0}^{n} f(x_i) A_i = Q \end{aligned}$$

That is, a quadrature rule is given by a weighted sum of values of f on the data pts, with weights $A_i = \int_a^b l_i(x)dx$. The weights depend only on a, b and $l_i(x)$. Since they do not depend on f, they are computed once and for all, then used for any f.

Precision of interpolatory rules

Recall the poly interpolation error formula

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} w(x)$$

Clearly, if f happens to be a poly of degree $\leq n$, the poly interpolant based on $n+1$ distinct pts is exact. Thus, the respective quadrature rule based on $n+1$ distinct pts is exact.

A quadrature rule which is exact on all polynomials of degree k or less and not exact on at least one poly of degree $k+1$ on one interval is said to have **polynomial degree** or **precision** k.

An interpolatory quadrature rule based on $n+1$ distinct pts has poly degree at least n.

Midpoint rule

Assume f is approximated by a constant poly $p_0(x) = f(m)$, where $m = \frac{a+b}{2}$ is the midpoint of $[a, b]$. Then

$$\begin{aligned} I &= \int_a^b f(x)dx \\ &\approx \int_a^b f(m)dx = f(m)(b-a) \end{aligned}$$

That is, I is approximated by the area of the rectangle under $p = f(m)$. If f happens to be a constant poly then the quadrature rule is exact. Therefore, the midpoint rule has poly degree at least 0 by construction.

Mean Value Theorem (MVT) for integrals
If f is cts on $[a, b]$ and g cts and one signed ($g \geq 0, g \leq 0$) then

$$\int_a^b f(x)g(x)dx = f(\eta) \int_a^b g(x)dx \text{ for some } \eta \in [a, b]$$

By using MVT, we can show

$$\int_a^b f(x)dx - (b-a)f(m) = f''(\eta)\frac{(b-a)^3}{24}$$

Thus, if f is a polynomial of degree ≤ 1, the midpt rule is exact. Therefore, the midpt rule has poly degree at least 1.

Take $f(x) = x^2, a = 0, b = 1$
$I = \int_a^b f(x)dx = \int_0^1 x^2 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$
But $Q_{midpt} = f(m)(b-a) = (\frac{1}{2})^2(1-0) = \frac{1}{4} \neq \frac{1}{3}$

Therefore, the midpt rule is not exact on some poly of degree 2 and some interval. Thus, the poly degree of the midpt rule is exactly 1.

Trapezoidal rule

Assume f is approximated by a poly $p_1(x)$ of degree ≤ 1 passing through pts $(a, f(a)), (b, f(b))$, i.e. $p_1(x) = \frac{x-b}{a-b}f(a) + \frac{x-a}{b-a}f(b)$. Then

$$I \approx \frac{b-a}{2}(f(a) + f(b))$$

By construction, the trapezoidal rule has poly degree at least 1. We can show

$$\int_a^b f(x)dx - \frac{b-a}{2}(f(a) + f(b)) = -f''(\eta)\frac{(b-a)^3}{12}$$

If f is a poly of degree ≤ 1, the trap rule is exact.

Take $f(x) = x^2, a = 0, b = 1$
$I = \int_a^b f(x)dx = \int_0^1 x^2 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$
But $Q_{trap} = \frac{1-0}{2}(f(0) + f(1)) = \frac{1}{2}(0+1) = \frac{1}{2} \neq \frac{1}{3}$

Therefore, the trap rule is not exact on some poly of degree 2 and some interval. Thus, the poly degree of the trap rule is exactly 1.

Alternative derivation of the trap rule

Consider the model interval $[-1, 1]$ and quadrature rule of the form $I = \int_{-1}^1 f(x)dx \approx A_0 f(-1) + A_1 f(1) = Q$. We will compute the weights A_0, A_1 using the method of undetermined coefficients and forcing the rule to be exact on $f(x) = 1, x$.

$f(x) = 1$
$\int_{-1}^1 1dx = x|_{-1}^1 = 2$
$A_0 \cdot 1 + A_1 \cdot 1 = A_0 f(-1) + A_1 f(1)$

$f(x) = x$
$\int_{-1}^1 xdx = \frac{x^2}{2}|_{-1}^1 = 0$
$A_0 \cdot (-1) + A_1 \cdot 1 = A_0 f(-1) + A_1 f(1)$

$$\begin{aligned} A_0 + A_1 &= 2 \\ -A_0 + A_1 &= 0 \end{aligned}$$

$\Rightarrow A_1 = 1, A_0 = 1$. Thus,

$$\begin{aligned} I &= \textstyle\int_{-1}^1 f(x)dx \approx f(-1) + f(1) \\ &= \tfrac{1-(-1)}{2}(f(-1) + f(1)) \end{aligned}$$

This is the trap rule on $[-1, 1]$.

Transforming Quadrature formulas to other intervals

Assume we have the quadrature rule

$$\int_\alpha^\beta g(t)dt \approx \sum_{i=0}^n A_i g(t_i)$$

and want an approximation to $\int_a^b f(x)dx$. Use the linear transformation

$$x = \frac{(b-a)t + a\beta - b\alpha}{\beta - \alpha}$$

Then

$$\int_a^b f(x)dx = \int_\alpha^\beta f\left(\frac{(b-a)t + a\beta - b\alpha}{\beta - \alpha}\right)\frac{b-a}{\beta - \alpha}dt$$

and

$$\int_a^b f(x)dx = \frac{b-a}{\beta - \alpha}\sum_{i=0}^n A_i f\left(\frac{(b-a)t_i + a\beta - b\alpha}{\beta - \alpha}\right)$$