Approximation and Integration

Given a complicated function $g(x)$, it is often desirable to approximate it by a simpler function $f(x)$. Often $g(x)$ is given in a tabular form as a set of values $(x_i, y_i), i = 0 \dots n$, where $y_i = g(x_i)$.

$f(x)$ **interpolates** $g(x)$ at pts x_i if $f(x_i) = g(x_i)$. f is called the **interpolant** of g. $f(x)$ **interpolates** data (x_i, y_i) if $f(x_i) = y_i$. f is called the **interpolant** of the data.

Examples of practical situations where approximation is needed:
- Evaluating g at intermediate points.
- Evaluating g'.
- Evaluating the integral of g.

Polynomial Approximation

Polynomials are often chosen to approximate complicated functions. Some of the reasons include:
1) Poly are easy to evaluate
2) Poly are easy to integrate
3) Poly are easy to differentiate
4) Poly provide good approx to arbitrary cts functions

Weierstrass Theorem:
If g is a cts function on $[a, b]$, then for every $\varepsilon > 0$ there exists a poly $p_n(x)$ of degree $n = n(\varepsilon)$ such that

$$\max_{x \in [a,b]} |g(x) - p_n(x)| < \varepsilon$$

Caveats:
1) n may be high
2) The method does not give a method to construct p_n
3) p_n may not be an interpolant

Monomial basis

Let $(x_i, y_i), i = 0, \dots, n$ be the given data
We want to construct a poly of degree at most n that interpolates the data.

Let $p_n(x) = \sum_{j=0}^{n} \alpha_j x^j$.
The interpolating equations
$p_n(x_i) = y_i, i = 0, \dots, n$ take the form of a linear system $B\alpha = y$, where

$$B = \begin{bmatrix} 1 & x_0 & \dots & x_0^n \\ 1 & x_1 & \dots & x_1^n \\ & & \dots & \\ 1 & x_n & \dots & x_n^n \end{bmatrix}$$

B is called the **Vandermonde** matrix and is ill-conditioned for large n. It can be shown that B is non-singular iff the pts x_i are distinct. In this case we can uniquely solve the equations and compute $\alpha_1, \dots, \alpha_n$

Lagrange basis

Instead of using $1, x, \dots, x^n$ as basis functions use $l_0(x), l_1(x), \dots, l_n(x)$ where $l_j(x)$, called **Lagrange basis** function, is a poly of degree n satisfying

$$l_j(x_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Then $p_n(x) = \sum_{j=0}^{n} \alpha_j l_j(x)$.
From our interp cond. we have
$p_n(x_i) = \sum_{j=0}^{n} \alpha_j l_j(x_i) = y_i$
$\Rightarrow \alpha_i \cdot 1 = y_i \Rightarrow \alpha_i = y_i$.
The linear system is $I\alpha = y, \alpha = y$.

Analytical form of $l_j(x)$

$$\begin{aligned} l_j(x) &= \frac{(x - x_0)(x - x_1) \dots (x - x_n)}{(x_j - x_0)(x_j - x_1) \dots (x_j - x_n)} \\ &= \Pi_{i=0, i \neq j}^{n} (x - xi) / \Pi_{i=0, i \neq j}^{n} (x_j - xi) \end{aligned}$$

Existence and Uniquesness Theorem:
Given data $(x_i, y_i), i = 0, \ldots n$, with x_i distinct, there exists a unique interplant poly of degree $\leq n$ which interpolates the data.

Notation: $spr\{s_0, s_1, \ldots, s_n\}$ is the smallest interval containing $s_0, s_1, \ldots, s_n$.

Error Theorem:
If g has $n+1$ cts derivatives and $p_n(x)$ is the poly interp of degree $\leq n$, interpolating g at $x_0, x_1, \ldots, x_n$ then for any x

$$g(x) - p_n(x) = \frac{g^{(n+1)}(\xi)}{(n+1)!} w(x),$$

where $w(x) = (x - x_0) \ldots (x - x_n)$ and ξ is an unknown pt in $spr\{x, x_0, x_1, \ldots, x_n\}$.

Remarks

1) Error of the interpolant is zero on x_i (expected)

2) ξ is unknown, changes with x, $g^{(n+1)}(\xi)$ unknown. But a bound on $|g^{(n+1)}(\xi)|$ may be possible to estimate

3) $w(x)$ can be calculated for a particular x. For arbitrary x in a certain interval, a bound on $|w(x)|$ is possible to estimate. Let $[a, b] = spr\{x_0, x_1, \ldots, x_n\}$. If $n \leq 2$, $w(x)$ is cubic, $w'(x)$ is quadratic. It is easy to find roots r_1, r_2.
$\max_{a \leq x \leq b} |w(x)|$
$= \max\{|w(x)|, x = a, x = b, x = r_1, x = r_2\}$

When we are given a function g to interpolate we cannot control the size of its $n+1$ derivative. But we may be able to control the size of $|w(x)|$ by choosing the x_i's so that $\max_{a \leq x \leq b} |w(x)|$ is minimized.

If the x_i's are equidistant $w(x)$ tends to become bigger towards the end pts and smaller towards the midpt $\frac{a+b}{2}$.

Certain choices for non-equidistant x_i even-up $w(x)$ along $[a, b]$ so that $\max |w(x)|$ is minimized.

Newton's basis Divided Differences

Another basis is
$1, (x - x_0), (x - x_0)(x - x_1), \ldots,$
$(x - x_0)(x - x_1) \ldots (x - x_{n-1})$
$b_j(x) = \Pi_{i=0}^{j-1} (x - x_i)$ (assume $b_0(x) = 1$)
Then
$p_n(x) = \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1)$
$+ \ldots + \alpha_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$

We define the divided differences (DD's) by
$p[x_i] \equiv y_i$
$p[x_i, x_{i+1}] \equiv \frac{y_{i+1} - y_i}{x_{i+1} - x_i}$
$p[x_i, \ldots, x_{i+k}] \equiv \frac{p[x_{i+1}, \ldots, x_{i+k}] - p[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$

It can be shown that $\alpha_i = p[x_0, \ldots, x_i]$.
We can find α_i by using a Newton Divided Difference (NDD) table.

Comparision of the 3 bases

- Monomials

$p_n(x)$ efficiently evaluated.
Easy to differentiate and integrate.
Need to often solve ill-conditioned system $B\alpha = y$.
Adding a piece of data requires recomputation of all α_i's.

- Lagrange

Easy to find α_i's.
$p_n(x)$ is messy to evaluate.
Messy to differentiate and integrate.
Adding a piece of data requires updating all of the $l_j(x)$'s.

- Newton

Relatively easy to find the α_i's.
$p_n(x)$ is easy to evaluate.
Messy to differentiate and integrate.
Adding a piece of data is easy.