Multiple Integrals

So far we have only considered 1D integrals. But we live in a multi-dimensional world and need to approximate quantities like:

$$\int\int_D f(x,y)dxdy \quad \text{and} \quad \int\int\int_V f(x,y,z)dxdydz.$$

This is in general a difficult problem, but some simple approaches can be derived from our 1D background work.

Consider the case of an iterated integral

$$\begin{aligned} I(f) &= \int\int_D f(x,y)dxdy \\ &= \int_a^b (\int_{g_1(x)}^{g_2(x)} f(x,y)dy)dx, g_2 \geq g_1. \end{aligned}$$

Let $\Phi(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)dy$. Then

$$I(f) = \int_a^b \Phi(x)dx.$$

and we can apply a quadrature rule to $I(f)$. For example

$$I(f) \approx Q(f) = \sum_{i=0}^{n} w_i \Phi(x_i),$$

where $\Phi(x_i) = \int_{g_1(x_i)}^{g_2(x_i)} f(x_i,y)dy$ which we can approximate by

$$\Phi(x_i) \approx \sum_{j=0}^{m} u_j f(x_i, y_j).$$

Hence,

$$I(f) \approx Q(f) = \sum_{i=1}^{n}\sum_{j=1}^{m} w_i u_j f(x_i, y_j).$$

For example, if we apply Compound Trapezoidal rule with m panels in the y-direction and n panels in the x-direction we get

$$Q(f) = \frac{h_x}{2}[\Phi(a) + 2\sum_{i=1}^{n-1}\Phi(x_i) + \Phi(b)],$$

where $h_x = \frac{b-a}{n}, x_i = a + ih_x$, and

$$\begin{aligned} \Phi(x_k) &= \frac{h_{yk}}{2}[f(x_k, g_1(x_k)) + 2\sum_{j=1}^{m-1} f(x_k, y_{kj}) \\ &+ f(x_k, g_2(x_k))], \end{aligned}$$

where $h_{yk} = \frac{g_2(x_k) - g_1(x_k)}{m}$, $y_{kj} = g_1(x_k) + jh_{yk}$.

Legendre polynomials

Legendre polynomials are a collection of polynomials $\{P_0(x), \ldots, P_n(x), \ldots, \}$ with the properties:

1. For each $n, P_n(x)$ is a polynomial of degree n.
2. $\int_{-1}^{1} P(x)P_n(x)dx = 0$ whenever $P(x)$ is a polynomial of degree less than n.

The first few Legendre polynomials are

$$P_0(x) = 1, P_1(x) = x, P_2(x) = x^2 - \frac{1}{3},$$
$$P_3(x) = x^3 - \frac{3}{5}x, P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

Theorem:
Suppose that $x_1, x_2, \ldots, x_n$ are the roots of the nth Legendre polynomial $P_n(x)$ and that for each $i = 1, 2, \ldots, n$, the numbers w_i are defined by

$$w_i = \int_{-1}^{1} \Pi_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} dx.$$

If $P(x)$ is any polynomial of degree less than $2n$, then

$$\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} w_i P(x_i).$$

Least squares approximation

Program A:
Input: data(x_i, y_i), $i = 0, \ldots, m$
Output: a function $f(x)$ which approximates data st $|f(x_i) - y_i|$ is "small" but does not necessarily match the data exactly.
Solution: force f to minimize $\sum_{i=0}^{m} w_i(f(x_i) - y_i)^2$ for some positive weights w_i.
f : **discrete least squares** approximation to the data.

Program B:
Input: a function $y(x)$ in $[a, b]$.
Output: a function $f(x)$ which approximates y st. $|f(x) - y(x)|$ is "small" for $x \in [a, b]$.
Solution: force f to minimize $\int_a^b w(x)(f(x) - y(x))^2 dx$ for some positive and cts weight function $w(x)$.
f : **continuous least squares** approximation to the data.

Inner products and norms of functions

Let f and g be cts functions in $[a, b]$.
Discrete inner product of f and g at distinct points x_i, $i = 0, \ldots, m$ wrt weights w_i, $i = 0, \ldots, m$

$$< f, g >= \sum_{i=0}^{m} w_i f(x_i) g(x_i).$$

Continuous inner product of f and g in $[a, b]$, wrt weight function $w(x)$

$$< f, g >= \int_a^b w(x) f(x) g(x) dx.$$

Norm of f: $\|f\| = \sqrt{< f, f >}$.

Therefore, the problem of constructing a least squares approximation $f(x)$ to a given set of data $\{(x_i, y_i)\}$ or to a given function $y(x)$ is to construct an $f(x)$ that minimizes the norm of the error $\|f - y\|$.

Linear independence and orthogonality of functions

The set of functions $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is **linearly independent** in $[a, b]$, if, whenever

$$c_0\phi_0(x) + \ldots + c_n\phi_n(x) = 0 \;\; \forall x \in [a, b]$$

then $c_0 = c_1 = \ldots = c_n = 0$.

Theorem: If $\phi_i(x)$ is a polynomial of degree (exactly) i, for $i = 0, \ldots, n$, then $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is linearly independent on any interval $[a, b]$.

Theorem: If $\{\phi_0, \phi_1, \ldots, \phi_n\}$ is a set of linearly independent polynomials of degree at most n, then any polynomial $p_n(x)$ of degree at most n can be uniquely represented as a linear combination of $\phi_0, \ldots, \phi_n$, $p_n(x) = \sum_{i=0}^{n} c_i\phi_i(x)$.

Orthogonal Polynomials

Two polynomials $p(x)$ and $q(x)$ are **orthogonal** wrt the inner product $<,>$, if

$$< p, q >= 0.$$

The set of polynomials $\{q_i(x)\}_{i=0}^n$ is **orthogonal** wrt inner products $<,>$, if

$$< q_i, q_j >= \begin{cases} 0 & \text{for } i \neq j \\ \neq 0 & \text{for } i = j. \end{cases}$$

The set of polynomials $\{q_i(x)\}_{i=0}^n$ is **orthonormal** wrt inner products $<,>$, if

$$< q_i, q_j >= \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j. \end{cases}$$

Theorem: An orthogonal (or orthonormal) set of polynomials is also linearly independent.

Chebychev polynomials are orthogonal wrt the continuous inner product $< p, q >= \int_{-1}^{1} w(x)p(x)q(x)dx$ with $w(x) = \frac{1}{\sqrt{1-x^2}}$.

Legendre polynomials are orthogonal wrt the continuous inner product $< p, q >= \int_{-1}^{1} w(x)p(x)q(x)dx$ with $w(x) = 1$.

Theorem: If $\{q_0, \ldots, q_n\}$ is an orthonormal set of polynomials on an interval $[a, b]$ with respect to the weight function w, then the least squares approximation to y on $[a, b]$ with respect to w is

$$p^*(x) = \sum_{j=0}^{n} c_j q_j(x),$$

where, for each $j = 0, \ldots, n$, $c_j =< q_j, y >$.

Three-term recurrence relation alogrithm

Output:
$\{q_i\}_{i=0}^n$ orthonormal poly wrt $<,>$
$\{s_i\}_{i=0}^n$ orthogonal poly wrt $<,>$

$s_0 = 1$, $\quad q_0 = \frac{s_0}{\|s_0\|}$
$\beta_1 = \frac{< x s_0, s_0 >}{< s_0, s_0 >}$
$s_1 = x - \beta_1$, $\quad q_1 = \frac{s_1}{\|s_1\|}$
for $i = 2$ to n do
  $\beta_i = \frac{< x s_{i-1}, s_{i-1} >}{< s_{i-1}, s_{i-1} >}$
  $\gamma_i = \frac{< x s_{i-2}, s_{i-1} >}{< s_{i-2}, s_{i-2} >}$
  $s_i = (x - \beta_i) s_{i-1} - \gamma_i s_{i-2}$
  $q_i = \frac{s_i}{\|s_i\|}$
end

This algo requires $2n + 1 \approx 2n = O(n)$ inner products to be computed and is the standard algorithm for constructing sets of orthogonal polynomials.

Constructing the least squares polynomial approximation

Input: data $(x_i, y_i), i = 0, \ldots, m$ or
a cts function $y(x)$ in the interval $[a, b]$
and its inner product(discrete or cts).
Output: $p^*(x)$, the least squares polynomial
of degree $\leq n \leq m$ approximation
to y wrt the inner product $<,>$.
Algorithm:

- Compute the $n + 1$ orthonormal polynomials $q_j, j = 0, \ldots, n$, through the three-term recurrence relation algorithm
- Compute $n + 1$ $< y, q_j >, j = 0, \ldots, n$
- Form $p^*(x) = \sum_{j=0}^{n} < y, q_j > q_j$.