Calculating Cubic Splines

Problem:

Given a set of $n+1$ data points $\{(x_i, y_i)\}_{i=0}^{n}$ or $\{(x_i, f(x_i))\}_{i=0}^{n}$, determine a smooth spline function $S(x)$ that interpolates the data, and where $S(x) = S_i(x)$ when $x \in [x_i, x_{i+1}]$.

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Define $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$, for $i = 0, 1, \ldots, n$. (The polynomial $S_n(x)$ is a "fictitious" piece that helps with the calculation of the polynomial coefficients.)

Step 1 – Determine the step sizes $\{h_i\}_{i=0}^{n-1}$

Calculate $h_i = x_{i+1} - x_i$, for $i = 0, 1, \ldots n-1$.

Step 2 – Determine $\{c_i\}_{i=0}^{n-1}$

We will have to set up and solve a dimension $(n+1) \times (n+1)$ linear system $\mathbf{M}\underline{c} = \underline{r}$. The system will depend on whether we are finding a **natural** spline or a **clamped** spline.

Natural splines

In this case, the linear system has the form:

$$\begin{bmatrix} \mathbf{1} & & & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & & \mathbf{1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \frac{3}{h_1}(y_2 - y_1) - \frac{3}{h_0}(y_1 - y_0) \\ \frac{3}{h_2}(y_3 - y_2) - \frac{3}{h_1}(y_2 - y_1) \\ \vdots \\ \frac{3}{h_{n-1}}(y_n - y_{n-1}) - \frac{3}{h_{n-2}}(y_{n-1} - y_{n-2}) \\ \mathbf{0} \end{bmatrix}$$

Solve this linear system for $\{c_i\}_{i=0}^{n}$ using Gaussian Elimination. Note: This can be reduced to a dimension $(n-1) \times (n-1)$ system because the first and last equations tell us that $c_0 = 0$ and $c_n = 0$.

Clamped splines

The first and last equations from the natural spline case change to account for the different boundary conditions.

$$\begin{bmatrix} \mathbf{2h_0} & \mathbf{h_0} & & & & \\ h_0 & 2(h_0+h_1) & h_1 & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & \mathbf{h_{n-1}} & \mathbf{2h_{n-1}} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \\ c_n \end{bmatrix} = \begin{bmatrix} \mathbf{\frac{3}{h_0}(y_1 - y_0) - 3f'(x_0)} \\ \frac{3}{h_1}(y_2 - y_1) - \frac{3}{h_0}(y_1 - y_0) \\ \frac{3}{h_2}(y_3 - y_2) - \frac{3}{h_1}(y_2 - y_1) \\ \vdots \\ \frac{3}{h_{n-1}}(y_n - y_{n-1}) - \frac{3}{h_{n-2}}(y_{n-1} - y_{n-2}) \\ \mathbf{3f'(x_n) - \frac{3}{h_{n-1}}(y_n - y_{n-1})} \end{bmatrix}$$

Solve this linear system for $\{c_i\}_{i=0}^{n}$ using Gaussian Elimination .

Step 3 – Determine $\{a_i\}_{i=0}^{n-1}$

Calculate $a_i = y_i, i = 0, 1, \ldots n-1$.

Step 4 – Determine $\{b_i\}_{i=0}^{n-1}$

Calculate $b_i = \dfrac{y_{i+1} - y_i}{h_i} - \dfrac{h_i}{3}(2c_i + c_{i+1}), i = 0, 1, \ldots n-1$.

Step 5 – Determine $\{d_i\}_{i=0}^{n-1}$

Calculate $d_i = \dfrac{c_{i+1} - c_i}{3h_i}, i = 0, 1, \ldots n-1$.

Done! Your spline has been calculated.