CSC 411: Introduction to Machine Learning

Lecture 5: Ensembles II

Mengye Ren and Matthew MacKay

University of Toronto

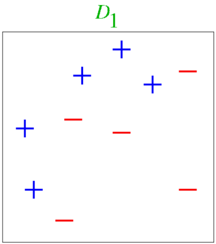
Boosting

- Recall that an *ensemble* is a set of predictors whose individual decisions are combined in some way to classify new examples.
- (Previous lecture) **Bagging**: Train classifiers independently on random subsets of the training data.
- (This lecture) **Boosting**: Train classifiers sequentially, each time focusing on training data points that were previously misclassified.
- Let us start with the concept of **weak learner/classifier** (or base classifiers).

Weak Learner/Classifier

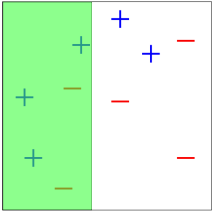
- (Informal) Weak learner is a learning algorithm that outputs a hypothesis (e.g., a classifier) that performs slightly better than chance, e.g., it predicts the correct label with probability 0.6.
- We are interested in weak learners that are *computationally* efficient.
  - Decision trees
  - Even simpler: **Decision Stump**: A decision tree with only a single split

[Formal definition of weak learnability has quantifies such as "for any distribution over data" and the requirement that its guarantee holds only probabilistically.]

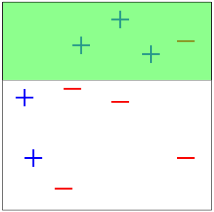


These weak classifiers, which are decision stumps, consist of the set of horizontal and vertical half spaces.

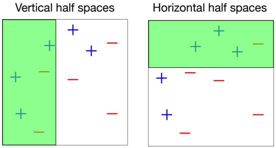
Vertical half spaces



Horizontal half spaces



Weak Classifiers



- A *single* weak classifier is not capable of making the training error very small. It only perform slightly better than chance, i.e., the error of classifier h according to the given weights $\mathbf{w} = (w_1, \ldots, w_N)$ (with $\sum_{i=1}^{N} w_i = 1$ and $w_i \geq 0$)

$$\text{err} = \sum_{i=1}^{N} w_i \mathbb{I}\{h(\mathbf{x}_i) \neq y_i\}$$

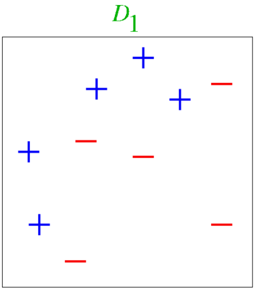
  is at most $\frac{1}{2} - \gamma$ for some $\gamma > 0$.
- Can we combine a set of weak classifiers in order to make a better ensemble of classifiers?
- Boosting: Train classifiers sequentially, each time focusing on training data points that were previously misclassified.

AdaBoost (Adaptive Boosting)

- Key steps of AdaBoost:
  1. At each iteration we re-weight the training samples by **assigning larger weights** to samples (i.e., data points) that were **classified incorrectly**.
  2. We train a new weak classifier based on the **re-weighted samples**.
  3. We **add this weak classifier to the ensemble** of classifiers. This is our new classifier.
  4. Weight each weak classifier in the ensemble with some weights.
  5. We repeat the process many times.
- The weak learner needs to minimize weighted error.
- AdaBoost reduces **bias** by making each classifier focus on previous mistakes.

AdaBoost Example

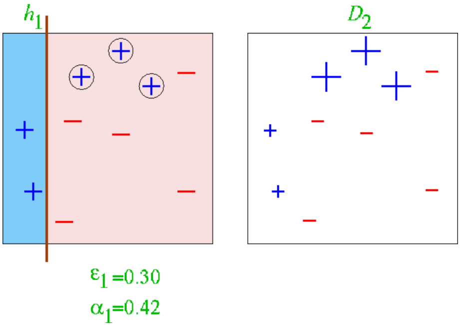
- Training data



[Slide credit: Verma & Thrun]

AdaBoost Example

- Round 1
- ϵ : *Trainingerror*, α: Weighting of the current tree.

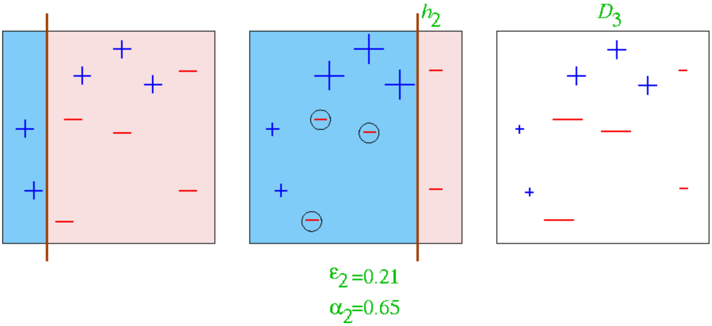


$$\mathbf{w} = \left(\frac{1}{10}, \ldots, \frac{1}{10}\right) \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_1 = \frac{\sum_{i=1}^{10} w_i \mathbb{I}\{h_1(\mathbf{x}^{(i)}) \neq t^{(i)}\}}{\sum_{i=1}^{N} w_i} = \frac{3}{10}$$

$$\Rightarrow \alpha_1 = \frac{1}{2} \log \frac{1 - \text{err}_1}{\text{err}_1} = \frac{1}{2} \log(\frac{1}{0.3} - 1) \approx 0.42 \Rightarrow H(\mathbf{x}) = \text{sign}\left(\alpha_1 h_1(\mathbf{x})\right)$$

AdaBoost Example

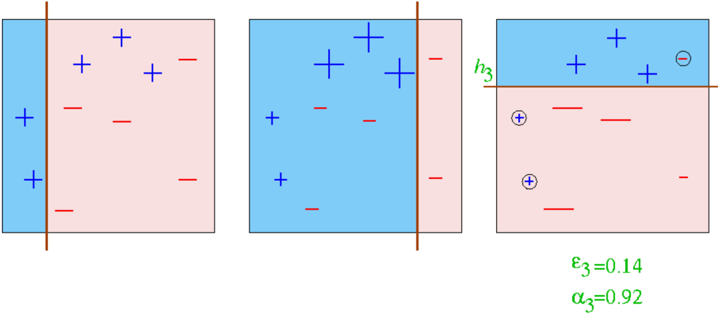
- Round 2



$$\mathbf{w} = \text{updated weights} \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_2 = \frac{\sum_{i=1}^{10} w_i \mathbb{I}\{h_2(\mathbf{x}^{(i)}) \neq t^{(i)}\}}{\sum_{i=1}^{N} w_i} = 0.21$$

$$\Rightarrow \alpha_2 = \frac{1}{2} \log \frac{1 - \text{err}_3}{\text{err}_3} = \frac{1}{2} \log(\frac{1}{0.21} - 1) \approx 0.66 \Rightarrow H(\mathbf{x}) = \text{sign}\left(\alpha_1 h_1(\mathbf{x}) + \alpha_2 h_2(\mathbf{x})\right)$$

- Round 3



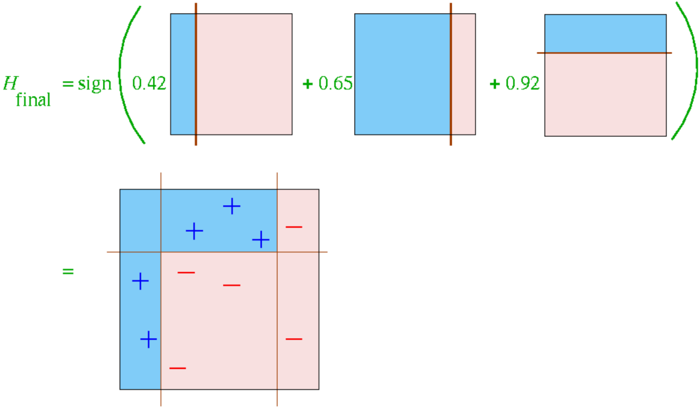
$$\mathbf{w} = \text{updated weights} \Rightarrow \text{Train a classifier (using } \mathbf{w}) \Rightarrow \text{err}_3 = \frac{\sum_{i=1}^{10} w_i \mathbb{I}\{h_3(\mathbf{x}^{(i)}) \neq t^{(i)}\}}{\sum_{i=1}^{N} w_i} = 0.14$$

$$\Rightarrow \alpha_3 = \frac{1}{2}\log\frac{1-\text{err}_3}{\text{err}_3} = \frac{1}{2}\log(\frac{1}{0.14}-1) \approx 0.91 \Rightarrow H(\mathbf{x}) = \text{sign}\left(\alpha_1 h_1(\mathbf{x}) + \alpha_2 h_2(\mathbf{x}) + \alpha_3 h_3(\mathbf{x})\right)$$

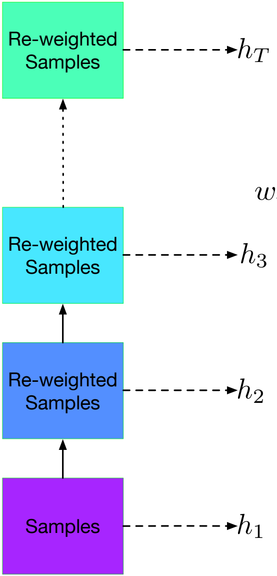
[Slide credit: Verma & Thrun]

AdaBoost Example

- Final classifier



[Slide credit: Verma & Thrun]



$$H(x) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(x)\right)$$

$$w_i \leftarrow w_i \exp\left(2\alpha_t \mathbb{I}\{h_t(\mathbf{x}^{(\mathbf{i})}) \neq \mathbf{t}^{(\mathbf{i})}\}\right)$$

$$\alpha_t = \frac{1}{2}\log\left(\frac{1 - \text{err}_t}{\text{err}_t}\right)$$

$$\text{err}_t = \frac{\sum_{i=1}^{N} w_i \mathbb{I}\{h_t(\mathbf{x}^{(\mathbf{i})} \neq \mathbf{t}^{(\mathbf{i})}\}}{\sum_{i=1}^{N} w_i}$$

AdaBoost Algorithm

- Input: Data $\mathcal{D}_N = \{\mathbf{x}^{(i)}, t^{(i)}\}_{i=1}^N$, weak classifier WeakLearn (a classification procedure that return a classifier from base hypothesis space $\mathcal{H}$ with $h : \mathbf{x} \to \{-1, +1\}$ for $h \in \mathcal{H}$), number of iterations T
- Output: Classifier $H(x)$
- Initialize sample weights: $w_i = \frac{1}{N}$ for $i = 1, \ldots, N$
- For $t = 1, \ldots, T$
  - Fit a classifier to data using weighted samples ($h_t \leftarrow WeakLearn(\mathcal{D}_N, \mathbf{w})$), e.g.,

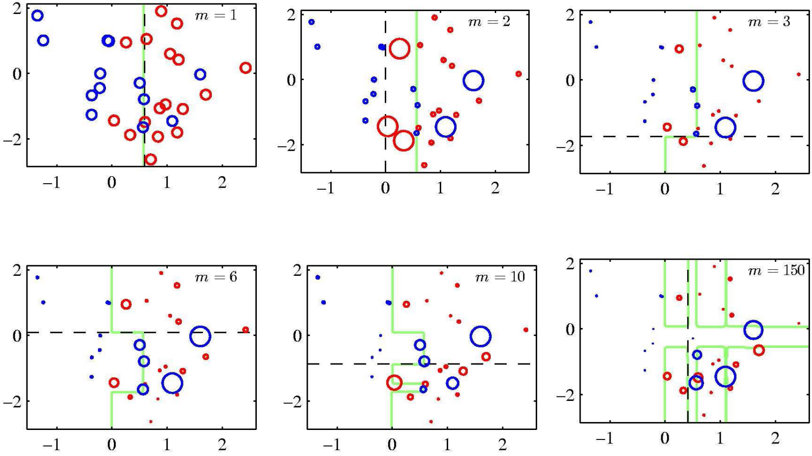
$$h_t \leftarrow \operatorname*{argmin}_{h \in \mathcal{H}} \sum_{i=1}^{N} w_i \mathbb{I}\{h(\mathbf{x}^{(i)}) \neq t^{(i)}\}$$

  - Compute weighted error $\text{err}_t = \frac{\sum_{i=1}^N w_i \mathbb{I}\{h_t(\mathbf{x}^{(i)}) \neq t^{(i)}\}}{\sum_{i=1}^N w_i}$
  - Compute classifier coefficient $\alpha_t = \frac{1}{2} \log \frac{1 - \text{err}_t}{\text{err}_t}$
  - Update data weights

$$w_i \leftarrow w_i \exp\left(-\alpha_t t^{(i)} h_t(\mathbf{x}^{(i)})\right) \left[\equiv w_i \exp\left(2\alpha_t \mathbb{I}\{h_t(\mathbf{x}^{(i)}) \neq t^{(i)}\}\right)\right]$$

- Return $H(\mathbf{x}) = \text{sign}\left(\sum_{t=1}^T \alpha_t h_t(\mathbf{x})\right)$

AdaBoost Example



- Each figure shows the number m of base learners trained so far, the decision of the most recent learner (dashed black), and the boundary of the ensemble (green)

AdaBoost Minimizes the Training Error

Theorem

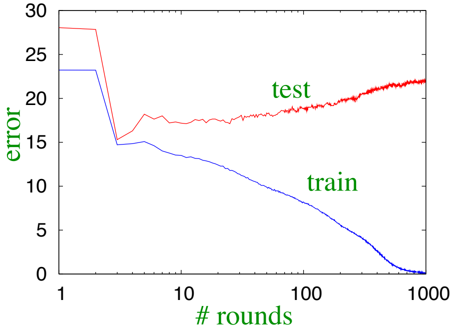
Assume that at each iteration of AdaBoost the WeakLearn returns a hypothesis with error $\text{err}_t \leq \frac{1}{2} - \gamma$ for all $t = 1, \ldots, T$ with $\gamma > 0$. The training error of the output hypothesis $H(\mathbf{x}) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t h_t(\mathbf{x})\right)$ is at most

$$L_N(H) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}\{H(\mathbf{x}^{(i)}) \neq t^{(i)})\} \leq \exp\left(-2\gamma^2 T\right).$$

- This is under the simplifying assumption that each weak learner is γ-better than a random predictor.
- Analyzing the convergence of AdaBoost is generally difficult.

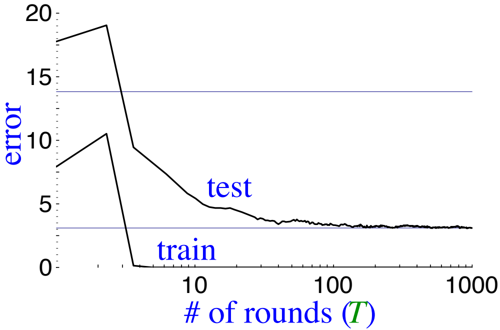
Generalization Error of AdaBoost

- AdaBoost's training error (loss) converges to zero. What about the test error of H?
- As we add more weak classifiers, the overall classifier H becomes more "complex".
- We expect more complex classifiers overfit.
- If one runs AdaBoost long enough, it can in fact overfit.



Generalization Error of AdaBoost

- But often it does not.
- Sometimes the test error decreases even after the training error is zero!

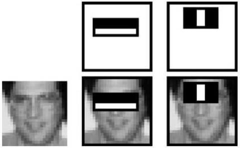


- How does that happen?
- We will provide an alternative viewpoint on AdaBoost later in the course.

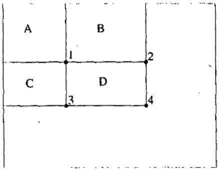
[Slide credit: Robert Shapire's Slides, `http://www.cs.princeton.edu/courses/archive/spring12/cos598A/schedule.html`]

AdaBoost for Face Recognition

- Viola and Jones created a very fast face detector that can be scanned across a large image to find the faces.

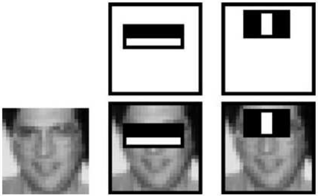


- The base classifier/weak learner just compares the total intensity in a rectangular filter.
- Integral image trick for evaluating the dot product very fast:

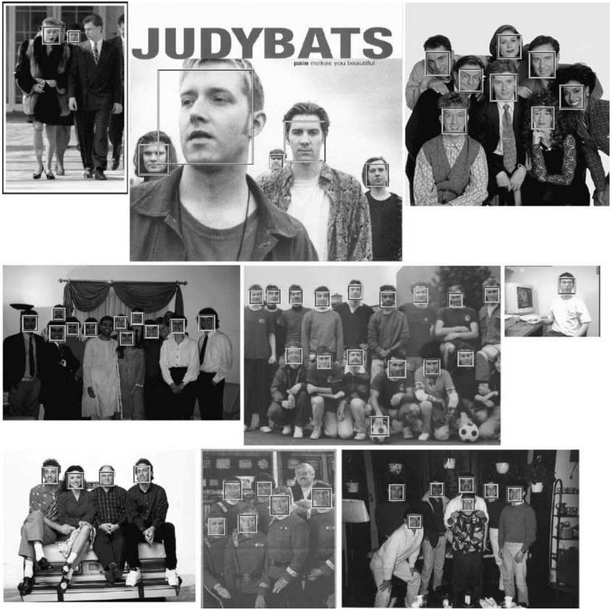


AdaBoost for Face Detection

- Famous application of boosting: detecting faces in images
- A few twists on standard algorithm
  - Pre-define weak classifiers, so optimization=selection
  - Smart way to do inference in real-time (in 2001 hardware)



AdaBoost Face Detection Results



Summary

- Boosting reduces bias by generating an ensemble of weak classifiers.
- Each classifier is trained to reduce errors of previous ensemble.
- It is quite resilient to overfitting, though it can overfit.
- We will later provide a loss minimization viewpoint to AdaBoost. It allows us to derive other boosting algorithms for regression, ranking, etc.

Ensembles Recap

- Ensembles combine classifiers to improve performance
- Boosting
  - Reduces bias
  - Increases variance (large ensemble can cause overfitting)
  - Sequential
  - High dependency between ensemble elements
- Bagging
  - Reduces variance (large ensemble can't cause overfitting)
  - Bias is not changed (much)
  - Parallel
  - Want to minimize correlation between ensemble elements.
- Next Lecture: Linear Regression