Abductive Expansion: Abductive Inference and the Process of Belief Change†

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ABSTRACT

Unlike the AGM framework of belief revision, where new information is added to the current belief state as it is (while attempting to preserve logical consistency), we model the following more natural proposal: *agents seek a justification (explanation) for the new information before "accepting" it and incorporate this justification into their current belief state together with the new information.* We model this proposal using the process of *abductive inference.* The resulting process is called *abductive expansion* and is characterised by a set of intuitive postulates. Constructions, similar in spirit to those proposed for AGM contraction and revision, are discussed.

1. Introduction

The enterprise of belief revision aims to model how an agent's beliefs change when presented with new information. Belief change is commonly represented as a transition from an initial belief (epistemic) state K to an updated belief state K'_α with respect to some new information (an *epistemic input*) α. Belief states (K, K'_α) are commonly represented by consistent sets of formulae while the epistemic input (α) is usually represented by a single formula.

The belief revision framework developed by Alchourrón, Gärdenfors and Makinson[1,3] (the AGM framework) identifies three types of change or transition: expansion (K^+_α) — introduction of a new belief α to K without retraction of any previous beliefs; contraction (K^-_α) — removal of a belief α from a belief state K without introduction of any new beliefs; and, revision (K^*_α) — introduction of a new belief α which may involve removal of existing beliefs from the belief state K. In the situation outlined above, the resulting belief state K'_α would be identified with one of K^+_α, K^-_α or K^*_α. An important consideration, central to the AGM framework, is the principle of minimal change — the new belief state should differ minimally (within reasonable limits) from the original belief state in achieving the desired change.*

The other topic with which we are concerned in this paper is that of abductive reasoning[2,7]. Abductive reasoning (or simply abduction) is a form of logical inference that attempts to derive "explanations" for a body of data. Often we are interested in determining a best "explanation" or abduction from among the alternatives. In fact, abduction is sometimes referred to as "inference to the best explanation"[6].

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*If belief states are represented using sets of formulae then minimality is usually taken with respect to set inclusion.

An abductive inference proceeds by proposing or generating hypotheses which would account for or "explain" a group of facts. For example, suppose we know that anyone suffering from measles develops a red rash on their skin. If you notice that someone has a red rash on their skin you might hypothesise that they are suffering from measles for this would account for your observation. There may of course be other explanations; the person in question may be suffering from some form of allergy, for instance.

The aim of this paper is to investigate certain aspects of the role of abductive reasoning within the process of belief revision. Our motivation stems from the following claim: often when we modify our belief state upon receipt of new information we do not just simply add the new information as it stands but, rather, first seek some type of justification and proceed by adding this justification to our belief state (and thus also obtaining belief in the new information).

Continuing our example, if someone comes to you with a rash on their skin you are more likely to add to your beliefs the fact that they are suffering from measles (or maybe some more plausible alternative) rather than simply adding the fact they have a red rash (especially if you are a doctor). In this paper we shall develop a framework incorporating the notions of abductive reasoning and belief revision in an attempt to model this claim.

A similar approach has been advocated by Kakas and Mancarella[8] who apply it to database updates. They argue that updating in this way may lead to the explicit representation of information that would otherwise be implicit and that could be lost. Furthermore, if a database is structured in a particular way then it may help to "transform" new information into a form reflecting this structure.

2. Preliminaries

We shall consider a fixed finite language $\mathcal{L}$ consisting of the standard logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and the symbol $\perp$ (falsum). We shall adopt the convention that lower case Greek letters $\alpha, \beta, \gamma \ldots$ denote formulae of $\mathcal{L}$ while upper case Greek letters $\Delta, \Gamma, \ldots$ denote sets of formulae. The underlying logic will be identified with its consequence operator $Cn : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ and is assumed to include classical propositional calculus. Moreover, it is assumd to satisfy compactness (*If* $\alpha \in Cn(\Gamma)$, *then* $\alpha \in Cn(\Gamma')$ *for some finite subset* $\Gamma' \subseteq \Gamma$) and the deduction theorem ($\beta \in Cn(\Gamma \cup \{\alpha\})$ *if and only if* $(\alpha \rightarrow \beta) \in Cn(\Gamma)$). A *theory* is a set of propositions closed under Cn. A set of propositions Γ is *consistent* if and only if for no proposition α do we have $\alpha \wedge \neg\alpha \in Cn(\Gamma)$.[†] We often write $\Gamma \vdash \alpha$ for $\alpha \in Cn(\Gamma)$.

The AGM postulates for expansion, contraction and revision attempt to characterise classes of functions for modelling the ways in which a rational agent alters its beliefs. In the AGM framework, belief states are commonly represented in terms of deductively closed sets of formulae called *belief sets* (denoted by upper case Roman letters H, K, $\ldots$) and epistemic inputs in terms of a formula α. We denote the set

[†]In the following, the inconsistent theory will be denoted by $\mathcal{L}$ or $K_{\perp}$ (the inconsistent belief set).

of all possible belief sets by **K**. In this presentation, as we shall soon see, our main concern is with the process of expansion so, for completeness, we reproduce the six basic expansion postulates[3]:

(K^+1)	K^+_α is a belief set	(closure)
(K^+2)	$\alpha \in K^+_\alpha$	(success)
(K^+3)	$K \subseteq K^+_\alpha$	(inclusion)
(K^+4)	If $\alpha \in K$, then $K^+_\alpha = K$	(vacuity)
(K^+5)	If $K \subseteq H$, then $K^+_\alpha \subseteq H^+_\alpha$	(monotonicity)
(K^+6)	For all belief sets K and all sentences α, K^+_α is the smallest belief set that satisfies (K^+1) — (K^+5)	(minimality)

An important representation theorem[1] characterises the process of expansion.

Theorem 1 *An expansion func. + satisfies (K^+1)—(K^+6) iff* $K^+_\alpha = Cn(K \cup \{\alpha\})$.

The definition of abduction that we shall adopt is similar to definitions commonly found in the literature[7,9].

Definition 1 *An abduction of a formula α with respect to a domain theory Γ is a formula β such that:*

(i) $\Gamma \cup \{\beta\} \vdash \alpha$*;*

(ii) $\Gamma \cup \{\beta\}$ *is consistent (i.e.,* $\Gamma \cup \{\beta\} \not\vdash \bot$*).*

For example, suppose $\Gamma = \{measles \rightarrow rash,\ allergic\_reaction \rightarrow rash,\ allergen \wedge allergy\_sufferer \rightarrow allergic\_reaction\ \}$ and $\alpha = rash$ then abductions include $measles$, $allergic\_reaction$, $measles \wedge allergic\_reaction$, $allergen \wedge allergy\_sufferer$.

Some definitions[7] consider abductions and the new information to be sets of formulae rather than single formulae as represented here by α and β. We shall concentrate on single formulae here as we are only interested in finitely expressible information and abductions and because it provides a nice correspondence with the AGM.

3. Abductive Expansion

Our aim is to model the claim that agents often seek some justification or "explanation" for newly acquired information and add this justification to their belief state together with the new information. We propose to model this process using the notion of abductive inference presented in Definition 1. Instead of merely adding an epistemic input (newly acquired information) to a belief state using the principle of minimal change, we shall present a framework where the agent adds an abduction of the epistemic input (α) to its current belief state (K). Since we shall be modelling belief states using belief sets, as in the AGM framework, the epistemic input will also appear in the new belief state (provided it is consistent with current beliefs).

From our definition of abduction we note that an abduction must be consistent with the domain theory, which in this case is represented by the initial belief state K. Therefore, since the formula we shall be attempting to add to our belief state is consistent with it, it does not really make sense at this stage to consider the process of revision and therefore we shall concentrate on the process of expansion. We shall refer to this process as the abductive expansion of K with respect to α and denote it $K^{\oplus}_{\alpha}$. In accord with our recent remarks, it can be defined as follows:[‡]

Definition 2 *$K^{\oplus}_{\alpha}$ is an abductive expansion of K with respect to α iff*

$$K^{\oplus}_{\alpha} = \begin{cases} Cn(K \cup \{\beta\}) & \textit{for some } \beta \in \mathcal{L} \textit{ such that:} \\ & \quad \textit{(i) } K \cup \{\beta\} \vdash \alpha \\ & \quad \textit{(ii) } K \cup \{\beta\} \nvdash \bot \\ & \\ K & \textit{if no such } \beta \textit{ exists}^{\S} \end{cases} \qquad \text{(Def AbExp)}$$

We begin by presenting a series of postulates to characterise this process.

3.1. Postulates for Abductive Expansion

An abductive expansion function $\oplus$ is assumed to be a function from pairs of beliefs sets and formulae to belief sets ($\oplus : \mathbf{K} \times \mathcal{L} \to \mathbf{K}$). It determines what belief state an agent would be in given that its current belief state is K and it has acquired new information α. This is captured by the following postulate:

(K$^{\oplus}$1) $K^{\oplus}_{\alpha}$ is a belief set (closure)

If the newly acquired information does not conflict with what is currently believed then any explanation or justification accepted should be for this new information. That is, the new information should be accepted in the resulting belief state in this case. Hence the following postulate:

(K$^{\oplus}$2) If $\neg\alpha \notin K$, then $\alpha \in K^{\oplus}_{\alpha}$ (limited success)

During the process of abductive expansion an agent is augmenting its currently held beliefs. Abductive expansion cannot result in the retraction of currently held beliefs thus leading to the following postulate.

(K$^{\oplus}$3) $K \subseteq K^{\oplus}_{\alpha}$ (inclusion)

In the limiting case where the new information conflicts with the currently held beliefs and it is thus not posible to find an abduction (i.e., a consistent "explanation") for it, no action is taken in order to maintain consistency. In this way, priority is given to established beliefs over new information (a skeptical agent!). A rational agent would refuse to accept the new information as it would lead to inconsistency (the situation is actually more complex and may be more appropriately handled by abductive revision which is briefly discussed in the conclusion).

(K$^{\oplus}$4) If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K$ (vacuity)

Whenever it is possible to find an explanation for new information the resulting belief state should remain consistent. That is, the explanation process should not

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[‡]Note that conditions (i) and (ii) correspond exactly to those in Definition 1.

[§]An alternative idea is to make $K^{\oplus}_{\alpha} = K_{\bot}$ if no β satisfying the stated criteria exists. This would give us a closer correspondence to AGM expansion in the case where $\neg\alpha \in K$. However, we have decided to place more emphasis on maintaining logical consistency rather than guaranteeing the inclusion of the epistemic input. This would also alter ($K^{\oplus}4$) to *If $\neg\alpha \in K$, then $K^{\oplus}_{\alpha} = K_{\bot}$* and guarantee success ($K^{\oplus}2$): $\alpha \in K^{\oplus}_{\alpha}$.

introduce any inconsistencies. This will be the case if the expanded belief state remains consistent with the new information.

(K$^{\oplus}$5) If $\neg\alpha \notin K$, then $\neg\alpha \notin K^{\oplus}_{\alpha}$ (consistency)

For reasons that will soon become apparent, we shall refer to (K$^{\oplus}$1) — (K$^{\oplus}$5) as the basic abductive expansion postulates. Another postulate that may be useful is:

(K$^{\oplus}$6) If $\vdash \alpha \leftrightarrow \beta$, then $K^{\oplus}_{\alpha} = K^{\oplus}_{\beta}$ (preservation)

This postulate expresses the fact that it is the content of the input rather than its linguistic formulation that is important in determining the change in belief state.¶ In effect, it guarantees a well-behaved abductive inference procedure in the sense that it will not act randomly in selecting which is the "best" abduction.

The following theorem tells us that postulates (K$^{\oplus}$1)—(K$^{\oplus}$5) capture the notion of abductive expansion that we desire.

Theorem 2 *The function $\oplus$ satisfies (K$^{\oplus}$1) — (K$^{\oplus}$5) iff $K^{\oplus}_{\alpha}$ is an abductive expansion of K with respect to α according to* (Def AbExp).

Note that only (K$^{\oplus}$1) — (K$^{\oplus}$5) are required to model the notion that we are trying to capture. Of course, since (K$^{\oplus}$1) — (K$^{\oplus}$5) satisfy the given definition of $K^{\oplus}_{\alpha}$ then (K$^{\oplus}$1) — (K$^{\oplus}$6) will also satisfy it. We prefer to consider the six postulates presented since, if only the first five are satisfied, then the abductive inference procedure may behave non-deterministically.

3.2. Constructions for Abductive Expansion

In the AGM framework for belief revision there are three major constructions investigated for the processes of contraction and revision: selection functions[1,3], Grove's system of spheres[5] and epistemic entrenchment[4]. For this section, we presume familiarity with these constructions. We shall continue our analysis of abductive expansion by investigating a construction based on selection functions.

If the epistemic input α is consistent with the current belief set K, the resulting (abductively) expanded belief set will be a superset of K containing α. Therefore, a choice has to be made from among the supersets of K that imply α. However, it is easily noted that every such superset can be represented as the intersection of a set of maximally consistent supersets of K that imply α. Accordingly we define:

Definition 3 *(Maximally Consistent Superset that Implies α)*
A set K' is a maximally consistent superset (of a set K) that implies α *if and only if*

- *(i)* $K \subseteq K'$
- *(ii)* $\alpha \in Cn(K')$
- *(iii)* $Cn(K') \neq K_{\perp}$
- *(iv) There is no $K'' \supset K'$ satisfying (i), (ii) and (iii)*

We use $K \top \alpha$ to denote the set of all the belief sets K' that are maximal consistent supersets of K that imply α. Note that if it were the case that $\neg\alpha \in K$ we obtain $K \top \alpha = \emptyset$. It is straightforward enough to show the following observation:

¶See also remarks in Gärdenfors[3] p. 55 regarding this property of contraction and revision.

Observation 3 *Any $K' \in K \top \alpha$ is a belief set.*

Various constructions can now be provided using the notion of a *selection function* γ. The most flexible idea would be for the selection function to choose a subset of the maximal consistent supersets implying α and take their intersection. Intuitively, γ selects the "best" elements from $K \top \alpha$. If $K^{\oplus}_{\alpha}$ is determined in this manner we call $\oplus$ a partial meet abductive expansion function.

Definition 4 (Def Part)

$$K^{\oplus}_{\alpha} = \begin{cases} \bigcap \gamma(K \top \alpha) & \textit{whenever } K \top \alpha \textit{ is nonempty} \\ K & \textit{otherwise} \end{cases}$$

Observation 4 $\bigcap \gamma(K \top \alpha)$ *is a belief set.*

The following theorem shows that this construction exactly characterises the postulates from Section 3.1 as desired.

Theorem 5 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a partial meet abductive expansion* if and only if $\oplus$ *satisfies postulates ($K^{\oplus}1$) — ($K^{\oplus}6$) for abductive expansion over K.*

By altering γ we, in effect, alter the selection mechanism that determines the "best" abduction for α with respect to K and therefore alter the type of expansion being performed. For example, one type of selection function γ_1 may give rise to $\oplus_1$ such that $measles \in K^{\oplus_1}_{rash}$ and $allergen, allergy\_sufferer \notin K^{\oplus_1}_{rash}$ while another, γ_2, gives rise to $\oplus_2$ such that $measles \notin K^{\oplus_2}_{rash}$ and $allergen, allergy\_sufferer \in K^{\oplus_2}_{rash}$.

Two special cases deserve mention. If $K^{\oplus}_{\alpha}$ is determined by a selection function γ such that $\gamma(K \top \alpha)$ is always a singleton for any K and α whenever $K \top \alpha \neq \emptyset$, then we call $\oplus$ a maxichoice abductive expansion function. Such a function would correspond to an abductive inference procedure that always hypothesises as much as possible (provided consistency is maintained) thus leading to logical omniscience. Instead of selecting just one element from $K \top \alpha$ we might, on the other hand, consider selecting them all and taking their intersection (assume a selection function γ such that $\gamma(K \top \alpha) = K \top \alpha$) whenever $K \top \alpha \neq \emptyset$. We call the resulting function $\oplus$ a full meet abductive expansion function. It would correspond to selecting the "trivial" abduction α for input α and thus implements the principle of minimal change as evidenced in the theorem below.

Lemma 6 *Maxi-choice and full meet abductive exp. funcs satisfy ($K^{\oplus}1$) — ($K^{\oplus}6$).*

Theorem 7 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a full meet abductive expansion function* iff $\oplus$ *satisfies postulates ($K^{+}1$) — ($K^{+}6$) for AGM expansion over K provided $\neg\alpha \notin K$.*

AGM expansion is then, in a certain sense, a limiting case of abductive expansion.

In analogy to the AGM, we can impose an ordering over elements of $K \top \alpha$. This will allow us to look at how the selection function γ determines the "best" elements of $K \top \alpha$. If the selection function γ picks out the elements of $K \top \alpha$ which are best according to some preference relation $\preceq$, then we say γ is relational over K. The most interesting case occurs when $\preceq$ is transitive (which is a minimal requirement of an ordering). In this case γ is called transitively relational. This is captured by the following two supplementary postulates for abductive expansion.

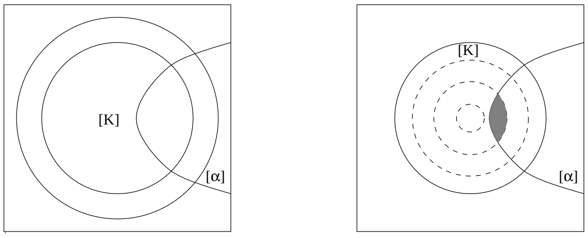

Figure 1: Grove system of spheres vs. internal system showing $[K_\alpha^\oplus]$ shaded.

(K$^\oplus$7) (a) $K_\alpha^\oplus \cap K_\beta^\oplus \subseteq K_{\alpha\vee\beta}^\oplus$ and (b) $K_\alpha^\oplus \subseteq Cn(K_{\alpha\vee\beta}^\oplus \cup \{\alpha\})$

(K$^\oplus$8) (a) *Either* $K_{\alpha\vee\beta}^\oplus \subseteq K_\alpha^\oplus$ *or* $K_{\alpha\vee\beta}^\oplus \subseteq K_\beta^\oplus$,
(b) *if* $\alpha \notin K_{\alpha\vee\beta}^\oplus$ *then* $K_{\alpha\vee\beta}^\oplus \subseteq K_\beta^\oplus$ and (c) *if* $\neg\alpha \notin K_{\alpha\vee\beta}^\oplus$ *then* $K_{\alpha\vee\beta}^\oplus \subseteq K_\alpha^\oplus$

These supplementary postulates may be motivated by their following consequence (in the presence of (K$^\oplus$1) — (K$^\oplus$6)): $K_{\alpha\vee\beta}^\oplus = K_\alpha^\oplus$ *or* $K_\beta^\oplus$ *or* $K_\alpha^\oplus \cap K_\beta^\oplus$.

Theorem 8 *Let $\oplus$ be an abductive expansion function. For every belief set K, $\oplus$ is a transitively relational partial meet abductive expansion* if and only if $\oplus$ *satisfies postulates (K$^\oplus$1) — (K$^\oplus$8) for abductive expansion over K.*

Transitively relational partial meet abductive expansion functions impose an ordering over maximally consistent supersets of K. Since maximally consistent sets can be viewed as "possible worlds", this ordering is essentially one over worlds consistent with current beliefs. This fact, not by coincidence, is similar to Grove's account of belief revision via a system of spheres[5]. Grove adopts an ordering over worlds inconsistent with current beliefs. Similarly, we suggest that a "spheres semantics" for abductive expansion can be given as captured by Figure 1. An abductive expansion can be determined by selecting a subset of worlds in $[K]\cap[\alpha]$ where $[K]$ represents the worlds consistent with K and $[\alpha]$ those consistent with α. Unlike Grove, we may view $[K]$ as an "internal" system of spheres and identify $[K_\alpha^\oplus]$ with those worlds in the intersection of $[\alpha]$ and the innermost internal sphere that $[\alpha]$ intersects.

As Grove's system of spheres represents an epistemic entrenchment relation, our internal system of spheres may be seen as representing an abductive entrenchment relation $\sqsubseteq$ over formulae of $\mathcal{L}$. We believe the following set of postulates characterise this relation (current beliefs are minimally entrenched while logical contradictions are maximally entrenched). For any $\alpha,\ \beta,\ \gamma \in \mathcal{L}$

(AE1)	If $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \gamma$ then $\alpha \sqsubseteq \gamma$	(transitivity)
(AE2)	If $\{\alpha\} \vdash \beta$ then $\beta \sqsubseteq \alpha$	(submission)
(AE3)	For any α and β, $\alpha \wedge \beta \sqsubseteq \alpha$ or $\alpha \wedge \beta \sqsubseteq \beta$	(reverse conjunctiveness)
(AE4)	When $K \neq K_\perp$, $\alpha \in K$ iff $\alpha \sqsubseteq \beta$ for all β	(minimality)
(AE5)	If $\alpha \sqsubseteq \beta$ for all α then $\vdash \neg\beta$	(maximality)

We can interpret $\alpha \sqsubseteq \beta$ as saying that it is at least as difficult to assume β as α in an abduction. For any abductive entrenchment $\sqsubseteq$ we denote by $\oplus_\sqsubseteq$ (the abductive expansion function determined by this ordering) the following operation:

(*C*$\oplus$) $\quad \beta \in K_\alpha^{\oplus_\sqsubseteq}$ *iff either* $\beta \in K$ *or both* $\neg\alpha \notin K$ *and* $\alpha \to \beta \sqsubset \alpha \to \neg\beta$

where $\sqsubset$ refers to the strict part of $\sqsubseteq$. By the principal clause of $(C\oplus)$, the new beliefs acquired through abduction of α are those sentences β such that the information in β relative to α is easier to assume than the information in $\neg\beta$ relative to α. We shall pursue this work further on a future occasion.

4. Conclusions and Future Work

We have characterised the process of abductive expansion in an attempt to model the claim that agent's first seek some explanation of newly acquired information before expanding their belief state. Our claim effectively results in a partial rejection of the notion of minimal change as it currently stands. Note that we are not advocating a return to a foundationalist approach to belief revision as there is no requirement that justifications be founded on a select set of "basic beliefs". Although our approach is not unrelated to a foundationalist approach our aims are not consistent with it.

Work has begun on investigating the process of abductive revision $K^{\otimes}_{\alpha}$ of K with respect to α defined using the following variant of the Levi identity: $K^{\otimes}_{\alpha} = (K^{\ominus}_{\neg\alpha})^{\oplus}_{\alpha}$. $K^{\ominus}_{\neg\alpha}$ represents some form of contraction while $K^{\oplus}_{\alpha}$ represents abductive expansion. $K^{\otimes}_{\alpha}$ represents the resulting (abductive) revision which would allow the incorporation of new information that does not agree with the current belief state.

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