Severe Withdrawal (and Recovery)

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Abstract. The problem of how to remove information from an agent's stock of beliefs is of paramount concern in the belief change literature. An inquiring agent may remove beliefs for a variety of reasons: a belief may be called into doubt or the agent may simply wish to entertain other possiblities. In the prominent AGM framework [1, 8] for belief change, upon which the work here is based, one of the three central operations, *contraction*, addresses this concern (the other two deal with the incorporation of new information). Makinson [23] has generalised this work by introducing the notion of a *withdrawal* operation.

Underlying the account proffered by AGM is the idea of *rational* belief change. A belief change operation should be guided by certain principles or integrity constraints in order to characterise change by a rational agent. One of the most noted principles within the context of AGM is the Principle of Informational Economy. However, adoption of this principle in its purest form has been rejected by AGM leading to a more relaxed interpretation. In this paper, we argue that this weakening of the Principle of Informational Economy suggests that it is only one of a number of principles which should be taken into account. Furthermore, this weakening points toward a Principle of Indifference. This motivates the introduction of a new belief removal operation that we call *severe withdrawal*. We provide rationality postulates for severe withdrawal and explore its relationship with AGM contraction. Moreover, we furnish possible worlds and epistemic entrenchment semantics for severe withdrawals.

Key words: AGM, belief change, belief contraction, epistemic entrenchment, severe withdrawal, systems of spheres.

1. Introduction

An inquiring agent must, among other things, deal with the problem of *belief change* (or belief revision) — how to modify its current epistemic state (stock of beliefs) in light of new information. One of the more popular accounts of belief change in recent times has been that introduced by Alchourrón, Gärdenfors and Makinson [1] (henceforth referred to as the AGM framework). The AGM framework for belief change distinguishes three types of transformations on epistemic states: *belief contraction* ($K \dot{-} \phi$) — removal of belief ϕ from epistemic state K without addition of any further beliefs; *belief expansion* ($K + \phi$) — addition of belief ϕ and its consequences without removal of any existing beliefs; and, *belief revision* ($K * \phi$) — addition of belief ϕ and its consequences with possible removal of existing beliefs in order to maintain consistency.

In this paper we are predominantly concerned with the process of belief removal — "contraction" and "withdrawal".[1] It is our aim here to introduce a new, principled, belief removal operation. Although distinct from AGM contraction, the two are related through their emergent revision behaviour. Moreover, when attention is restricted to all functions related in this way and satisfying certain intuitive principles, we find that these two proposals lie at opposite ends of the spectrum with respect to their degree of belief removal measured in terms of set-theoretic inclusion. The alternative proposal introduced here is contrasted with AGM contraction which can be seen as a point of reference in helping to understand the vagaries of this new approach. In the AGM vein, rationality postulates are provided for our proposal and two constructions of central importance to the AGM framework — systems of spheres and epistemic entrenchment — are adapted and used to further promote this comparison.

There are a number of reasons why an inquiring agent would be interested in removing beliefs from its current epistemic state. If the agent finds itself in an inconsistent state — believing contradictory information — then it can give up certain beliefs in an attempt to regain consistency.[2] On the other hand, an agent may want to suspend belief in a particular proposition because it no longer has any confidence in that proposition or simply because it would like to consider other possibilities. In either case, the overriding concern is that the agent no longer include the proposition in question among its beliefs. Moreover, if one subscribes to Levi's *Commensurability Thesis* [18, p. 65] which states that any reasonable transition between two epistemic states can be achieved through a sequence of expansions and contractions, then the importance of contraction is clearly evident.[3]

Principally, we are concerned with characterising that belief change undergone by those agents which act in accord with certain principles or "integrity constraints" commonly referred to as *rationality criteria* (see also Gärdenfors and Rott [11, p. 38]). Arguably the most well known of these criteria (especially in the context of the AGM framework) is the *Principle of Informational Economy* [8] which we shall present here in slightly more general guise as the *Principle of Economy*:[4]

- *The Principle of Economy:*

  Keep loss to a minimum.

This principle has, in fact, become largely synonymous with the AGM framework. A special instance of this constraint, where loss is measured in terms of set-theoretic inclusion (of epistemic states), is known as the *Principle of Conservatism* [16]. It can be considered the starting point for the AGM account of contraction; the motivating concern underlying the AGM notion of "maxichoice contraction" and the pathway to that of "partial meet contraction" [1].[5] An important point to note is that such a comparison, on the basis of set-theoretic inclusion, presupposes the association of a positive value of utility with every single item of belief. The Principle of (Informational) Economy (and consequently that of Conservatism) is a restricted case of the *Principle of Minimal Change* [16] which states that addition as well as loss should be kept to a minimum.

It is our contention here that the Principle of Economy has been severely compromised in the AGM framework. In its purest form, as the Principle of Conservatism, it has been shown to lead to undesirable consequences when applied to logically closed belief sets [2].[6] As a result its imposition is effected to a much less stringent degree. We claim that this principle is not, in fact, an overriding criterion but, instead, must be applied in combination with other, equally important, principles in order to obtain an intuitively satisfactory account of belief change. Moreover, these principles are in a state of tension with respect to each other (i.e., they have conflicting concerns). In this paper we advocate, in particular, *Principles of Indifference and Preference*. Briefly, taken together they state that an object held in equal or higher regard than another should be treated equally or more favourably than the latter. In fact, we argue that AGM contraction does embrace these principles to a limited extent. However, this partial adoption does not appear to be clearly motivated or even justified. Therefore, we propose a stricter adherence to the Principles of Indifference and Preference. As a result of this change in view, we propose a new form of contraction differing from that put forward by AGM. Moreover, the rationality criteria proposed are equally applicable to the two constructive modellings that we investigate here — *systems of spheres* and *epistemic entrenchment orderings*. Interestingly enough, this new contraction operation does not affect the AGM belief *revision* operation.

Let us return our focus of attention to the AGM development of belief contraction. Applying the Principle of Economy in the form of the Principle of Conservatism, it was at first suggested that the contraction of a belief set K by a sentence ϕ could be achieved by selecting some maximal subset of K

that does not imply ϕ. As mentioned above, this proposal was immediately abandoned as it has undesirable consequences. Instead, a selection function γ was applied to the set of all such maximal non-implying subsets, $K\bot\phi$[7], in order to select a set of the "best" elements which are then intersected to obtain a *partial meet contraction function* ($K \dot{-} \phi = \bigcap \gamma(K\bot\phi)$). It should be noted that the selection function γ is defined for all sentences ϕ but with K held fixed (i.e., for some belief set K, γ may take as an argument $K\bot\phi$ for any $\phi \in \mathcal{L}$). It is clear that this development leads away from conservatism. Certain objects are under scrutiny. A mechanism is used to discriminate among them although it may not be possible to distinguish some apart and so these are all retained and processed together. This leads us to formulate Indifference as follows:

- *The Principle of Indifference*

  Objects held in equal regard should be treated equally.

The situation in which all maximal non-implying subsets are held in equal regard is an AGM full meet contraction[8] which stands at the opposite end of the spectrum to AGM maxichoice contraction.

In settling on partial meet we realize that the Principle of Economy and the Principle of Indifference are in a state of tension with respect to one another; Economy advocates the selection of a single element from $K\bot\phi$ while Indiffence recommends to give up more than necessary if the selection mechanism does not single out a unique "best" solution. Both of these principles figure in the rationale behind the choice of "best" elements implicitly adopted in partial meet contraction. We shall extend this strategy by accepting the following, intuitively appealing, principle:

- *The Principle of Strict Preference*

  Objects held in higher regard should be afforded a more favourable treatment.

Taken together with the Principle of Indifference, this principle can be seen as advancing the following rather general principle:

- *The Principle of Weak Preference*

  If one object is held in equal or higher regard than another, the former should be treated no worse than the latter.

Such a principle can already be seen to be at work in AGM partial meet contraction with "importance" being judged through the selection function γ. However, consideration is restricted to the elements of $K\bot\phi$ for a particular sentence $\phi \in \mathcal{L}$. Our aim here is to emphasise an alternative to AGM contraction which we believe adheres more faithfully to the Principles of Indifference and Preference. However, adopting the AGM format for belief change processes (i.e., epistemic inputs as sentences from a suitable object language and epistemic states as sets of sentences that are deductively closed under some consequence relation) allows us to effect a straightforward comparison of the two proposals.

In the following section we outline some technical preliminaries. In sections 3 and 4 we present an intuitive overview of two important constructive modellings for AGM belief contraction. We describe how they fail to live up to the requirements demanded by the Principles of Indifference and Preference and outline an approach that resolutely favours these principles over the Principle of Economy. A common method of presenting AGM contraction operations is through rationality postulates which we survey in section 5 and contrast, in section 6, with rationality postulates for the belief removal operation advocated in the present paper. The relationship between AGM contraction operations and our account of belief removal is more directly addressed in section 7. In sections 8 through 11 we return to the constructive modellings discussed in sections 3 and 4, investigating the technical aspects of their application in our belief removal operation. This leads us to an investigation of the relationship between the two constructive modellings adopted here — systems of spheres and epistemic entrenchment — in section 12. We conclude with a discussion of the insights stemming from our approach, its relationship to other work in the literature (section 13) and, in section 14, a summary of the contributions made here.

2. Technical Preliminaries

Throughout this paper we assume a fixed propositional language $\mathcal{L}$ with countably many propositional symbols. We assume that $\mathcal{L}$ avails of the standard logical connectives, namely $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$, together with the propositional constants $\top$ (truth) and $\bot$ (falsum). The underlying logic will be identified with its *consequence operator* $Cn : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ which is assumed to satisfy the following properties.

$\Gamma \subseteq Cn(\Gamma)$	(Inclusion)
If $\Gamma \subseteq \Delta$, then $Cn(\Gamma) \subseteq Cn(\Delta)$	(Monotonicity)
$Cn(\Gamma) = Cn(Cn(\Gamma))$	(Iteration)
If ϕ can be derived from Γ by classical truth-functional logic, then $\phi \in Cn(\Gamma)$	(Supraclassicality)
$\psi \in Cn(\Gamma \cup \{\phi\})$ if and only if $(\phi \rightarrow \psi) \in Cn(\Gamma)$	(Deduction)
If $\phi \in Cn(\Gamma)$, then $\phi \in Cn(\Gamma')$ for some finite subset $\Gamma' \subseteq \Gamma$	(Compactness)

We often write $\Gamma \vdash \phi$ to mean $\phi \in Cn(\Gamma)$ and $\vdash \phi$ for $\emptyset \vdash \phi$.

We refer to any set of sentences K in $\mathcal{L}$ as a *belief set* or *theory* if K is closed under *Cn* (i.e., $K = Cn(K)$). One special belief set is the absurd belief set $K_{\perp}$ containing all sentences in $\mathcal{L}$. A belief set K is *consistent* in $\mathcal{L}$ if and only if it does not contain sentences ϕ and $\neg\phi$ for any $\phi \in \mathcal{L}$, i.e., if K does not equal $K_{\perp}$. A belief set is *complete* in $\mathcal{L}$ if either $\phi \in K$ or $\neg\phi \in K$ for every $\phi \in \mathcal{L}$. The set of all belief sets is denoted $\mathcal{K}$. We adopt the convention of denoting sentences by lower case Greek letters $\phi,\ \psi,\ \ldots$ and sets of sentences by upper case Roman letters $H,\ K,\ \ldots$.

3. Sphere-Based Withdrawal

An interesting way of viewing the process of belief change is in terms of possible worlds. A construction in this vein, specifically focussed on the AGM framework, has been proposed by Grove [12] who adapted Lewis' [21] possible worlds modelling for counterfactual conditionals. This approach possesses a highly intuitive appeal through the pictorial representation by systems of spheres.[9] In this section we concentrate on motivating our approach through this intuition, deferring the main technical details to section 8.

Grove [12] characterises the current beliefs of an agent by the collection of those possible worlds that are consistent with the agent's beliefs. But this is not the entire representation of an epistemic state. The remaining worlds — those inconsistent with the agent's current beliefs — are grouped around this core collection in decreasing order of plausibility. This results in a system of spheres centred on the set of worlds consistent with the agent's beliefs. Change in belief involves the determination of those worlds characterising the agent's new beliefs and is guided by the preference ordering over worlds.

More specifically, we denote the possible worlds consistent with a set of sentences K by $[K]$ and the set of all possible worlds by $\mathcal{W}$. We also adopt the shorthand $[\phi]$ for $[\{\phi\}]$. A *sphere* is simply a set of possible worlds $X \subseteq \mathcal{W}$. A *system of spheres centred on* $[K]$ is a set of nested spheres (in the sense of set inclusion) in which the smallest or innermost sphere is $[K]$ and the outermost sphere is $\mathcal{W}$. This is a generalisation of Lewis [21] whose systems of spheres are centred on a single world $w \in \mathcal{W}$ (the actual world) if we allow ourselves to neglect the fact that Lewis does not require $\mathcal{W}$ to be an element of every system of spheres. Essentially, a system of spheres centred on $[K]$ orders those worlds inconsistent with the agent's epistemic state K. Intuitively, the agent believes the actual world to be one of the K-worlds but does not have sufficient information to establish which one. However, the agent may be mistaken, in which case it believes that the actual world is most likely to be one of those in the next greater sphere and so on. As such, a system of spheres can be considered an ordering of *plausibility* over worlds;

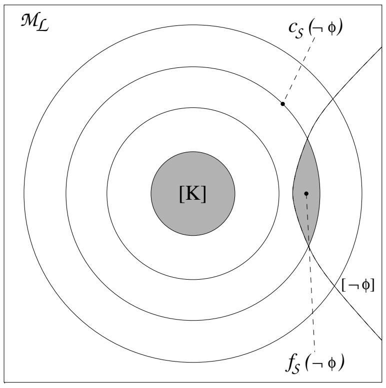

Figure 1. Sphere semantics for AGM belief contraction showing $[K \dot{-} \phi]$ shaded.

the more plausible worlds lying further towards the centre of the system of spheres.

This ordering provides us with a powerful tool for investigating the process of belief change. In this paper we concern ourselves with the operation of belief contraction. That is, the situation in which an agent wishes to suspend one of its beliefs. In this scenario, information is being removed, opening up more possibilities. In other words, the agent's candidate worlds increase; more worlds being added to $[K]$. In order to suspend belief in a sentence ϕ the agent must have some candidate worlds in which ϕ is false and therefore, considering the principal case in which ϕ is initially believed ($\phi \in K$), it must at least introduce some $\neg\phi$-worlds into $[K]$. The AGM approach to this problem is motivated to a large extent by the Principle of Informational Econcomy. Accordingly, simply the closest $\neg\phi$-worlds — those in the smallest sphere containing $\neg\phi$-worlds — are added to $[K]$. This situation is illustrated in Figure 1. If by $f_{\mathcal{S}}(\phi)$ we denote the ϕ-worlds closest to K and we introduce a function th that returns the belief set corresponding to a set of worlds, then we have the following method for defining an AGM contraction function $\dot{-}$ from a system of spheres:

(Def $\dot{-}$ from $\mathcal{S}$) $\quad K \dot{-} \phi = th([K] \cup f_{\mathcal{S}}(\neg\phi))$

It will also be convenient to refer to the smallest sphere intersecting ϕ which we denote by $c_{\mathcal{S}}(\phi)$. Then $f_{\mathcal{S}}(\phi)$ is given by $c_{\mathcal{S}}(\phi) \cap [\phi]$.

Now, if one were to apply the Principle of Informational Economy in its unadulterated form (i.e., Conservatism), then the aim of contraction — removal of a belief ϕ from epistemic state K — would be achieved through the addition of *a single* $\neg\phi$-world to $[K]$ rather than a number of $\neg\phi$-worlds as depicted in Figure 1. This form of contraction corresponds to *maxichoice contraction* in the AGM literature [1], i.e., the idea of taking belief contraction of K by ϕ to be some maximal subset of K that fails to imply ϕ.[10] However, this proposal has been shown to possess a number of drawbacks.

Foremost among these is the fact that any revision function defined from such a contraction function via the Levi Identity would always lead to a complete theory, i.e., $Cn\,(K \dot{-} \phi \cup \{\neg\phi\})$ is *maximally* consistent. This indicates that too little information is being removed. As a result, the Principle of Informational Economy is imposed on a much weaker level as indicated above. Instead of including only one $\neg\phi$-world in contraction, AGM incorporate a number of $\neg\phi$-worlds — those held to be most plausible — into the agent's epistemic state. However, none of the rationality postulates mentioned thus far specify precisely how to deal with worlds that are equally preferred by the agent. As a remedy, we suggest the employment of the Principle of Indifference. As we have seen, AGM have gone part of the way to adopting such a principle. However, they limit their embracement of such a strategy to the area covered by $\neg\phi$-worlds only. Presumably, this is due to a desire to remain as faithful to the Principle of Informational Economy as possible, despite its recognised shortcomings. Yet the Principle of Informational Economy has been compromised and its relevance called into question. We propose to place still less emphasis on its application and subordinate it to the Principle of Indifference.

Another principle, relating to the preference structure supplied by the sphere modelling, that we suggest to respect is the Principle of Strict Preference. According to this principle, worlds considered more plausible should be given more favourable treatment. When contracting its belief set with respect to ϕ, the agent must at least include some (one, at any rate) $\neg\phi$-world into its epistemic state. But the aforementioned principles, as applied to possible worlds and systems of spheres, advocate that any ϕ-worlds just as plausible as the innermost $\neg\phi$-worlds should be included also. Thus, together they sanction the following specialisation of the Principle of Weak Preference:

> *If one world is considered at least as plausible as another, then the former should be admitted in the agent's epistemic state if the latter is.*

The Principle of Informational Economy, in a weak form, can be viewed as limiting the extent of change to that sphere containing the closest $\neg\phi$-worlds and not beyond. The Principle of Weak Preference determines which worlds inside this limited region should be included in the new epistemic state. Without any further restrictions it suggests that all worlds inside this region should form part of the contracted epistemic state. In a way, even AGM appeal to this principle. There, however, the principle is only applied relative to $\neg\phi$-worlds, not all worlds in $\mathcal{W}$. However, no principle authorising a restricted imposition of this principle is established. The new situation is illustrated in Figure 2. The agent has determined a preference over worlds and does not prefer the (closest) $\neg\phi$-worlds over the (closer) ϕ-worlds just because it is giving up belief in ϕ. Its preferences are established prior to the change and we assume that there is no reason to alter them in light of the new information (epistemic input). It is for this reason that the Principle of Conservatism (the Principle

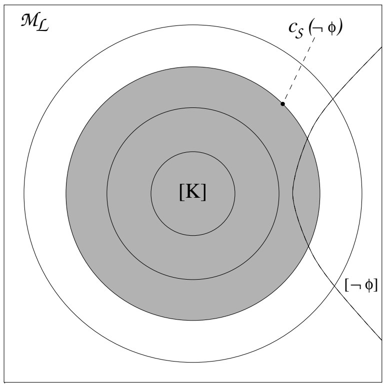

Figure 2. Sphere semantics for severe withdrawal showing $[K \ddot{-} \phi]$ shaded.

of Informational Economy in its pure form) must give way. We shall refer to this type of belief removal as *severe withdrawal*.

Denoting by $c_\mathcal{S}(\phi)$ the closest sphere to K containing ϕ-worlds, as mentioned above, we obtain a severe withdrawal function as follows:

(Def $\ddot{-}$ from $\mathcal{S}$) $\quad K \ddot{-} \phi = th(c_\mathcal{S}(\neg\phi))$

It is the study of this class of functions to which we devote ourselves here. When considered in a common setting, this form of belief removal has been independently advocated by Levi [20] who refers to such functions as *mild contractions*. Levi argues for mild contractions in terms of an information theoretic argument. We shall return to a consideration of Levi's arguments in the discussion (in Section 13).[11]

4. Entrenchment-Based Withdrawals

Lewis [21, Section 2.5] was perhaps the first to realise that a total ordering over possible worlds could be rephrased as a total ordering over the sentences of a language. Grove [12] provides such an ordering based on systems of spheres centred on $[K]$. Gärdenfors and Makinson [9] also introduce an ordering over sentences known as an *epistemic entrenchment*.[12] Intuitively, an epistemic entrenchment relation $\leq$ is an ordering over the agent's beliefs which reflects the plausibilities or degrees of retractability from a given belief state K. The relation $\phi \leq \psi$ can be read as "it is at least as hard to discard ψ than it is to discard ϕ." Epistemic entrenchment relations are thought of as satisfying a number of structural constraints which we need present only in Section 11.

Now let a relation $\leq$ of epistemic entrenchment be given, and let $<$ be its asymmetric part. Then the contraction based on $<$ as suggested by Gärdenfors and Makinson [9, (C-) condition)] is as follows.

$$(\text{Def } \dot{-} \text{ from } \leq) \qquad K \dot{-} \phi = \begin{cases} K \cap \{\psi : \phi < \phi \vee \psi\} & \text{if } \phi \in K \text{ and } \not\vdash \phi \\ K & \text{otherwise} \end{cases}$$

Although Gärdenfors and Makinson [9, p. 89] offer a motivation for this definition, it is rather hard to understand. Besides, as Gärdenfors and Makinson point out, their argument in support of (Def $\dot{-}$ from $\leq$) depends on the controversial postulate of recovery which we will discuss below.

In contraction, a basic idea seems to be that less epistemically entrenched sentences are to be given up in favour of more entrenched sentences [8, pp. 17–18, 75, 87]. Such an interpretation is vaguely reminiscent of an imposition of the Principle of Preference. In a related fashion, a more straightforward way of using $\leq$ was aired by Rott [29] (also compare Gärdenfors and Rott [11, p. 73]):

$$(\text{Def } \ddot{-} \text{ from } \leq) \qquad K \ddot{-} \phi = \begin{cases} K \cap \{\psi : \phi < \psi\} & \text{if } \phi \in K \text{ and } \not\vdash \phi \\ K & \text{otherwise} \end{cases}$$

We shall see that condition (Def $\ddot{-}$ from $\leq$) can in fact be used in both directions. On the reverse reading, ϕ is epistemically less entrenched then ψ, exactly when the successful removal of ϕ from epistemic state K results in the retention of ψ.

As in the case of Grovean sphere-based contractions (alias AGM contractions), the pure idea of minimising the amount of information lost is compromised in GM entrenchment-based contractions determined by (Def $\dot{-}$ from $\leq$). In general, the result of an entrenchment contraction is not a maximal subset of the theory K that does not entail ϕ.[13] If ϕ is in K, then every such maximal non-implying subset includes either $\phi \vee \psi$ or $\phi \vee \neg\psi$; if it did not, then it would not be maximal by disjunctive reasoning (which follows from our assumptions for *Cn*). However, it need not be the case that either $\phi < \phi \vee (\phi \vee \psi)$ or $\phi < \phi \vee (\phi \vee \neg\psi)$. The reason for this is that there may be *ties* in the plausibility of beliefs — just as there were ties in the plausibility of models. In the technical framework used for relations of epistemic entrenchment, the situation that neither $\phi \vee \psi$ nor $\phi \vee \neg\psi$ is in the contraction of K with respect to ϕ arises just in case $\phi \vee \psi$ is as equally plausible or entrenched as $\phi \vee \neg\psi$. So Gärdenfors and Makinson are ready to let the "Principle of Indifference" override the Principle of Minimal Change at least as far as disjunctions of ϕ and some other sentence are concerned. However, it is generally not the case that if ψ remains untouched in $K \dot{-} \phi$ and χ is equally or more entrenched than ψ, then χ remains untouched in $K \dot{-} \phi$ as well. Thus the Principles of Indifference and Preference are violated. We shall fully install these principles for the entrenchment-based removal of beliefs by endorsing (Def $\ddot{-}$ from $\leq$) in this paper. According to this definition, only preferences matter, the content of the beliefs remains totally disregarded.

5. The AGM Postulates for Contraction

In the preceding two sections we have discussed, from an intuitive standpoint, two constructive modellings for AGM belief change. These modellings are more often characterised by *rationality postulates* which specify axiomatic constraints that should be satisfied by any contraction operator of that particular type. As with the constructive modellings, they are guided by the rationality criteria outlined in the introduction. The following postulates are those for AGM contraction over a belief set K.

($\dot{-}$1) $K \dot{-} \phi = Cn\,(K \dot{-} \phi)$

($\dot{-}$2) $K \dot{-} \phi \subseteq K$

($\dot{-}$3) If $\phi \notin K$, then $K \subseteq K \dot{-} \phi$

($\dot{-}$4) If $\nvdash \phi$, then $\phi \notin K \dot{-} \phi$

($\dot{-}$5) $K \subseteq Cn\,((K \dot{-} \phi) \cup \{\phi\})$

($\dot{-}$6) If $Cn\,(\phi) = Cn\,(\psi)$, then $K \dot{-} \phi = K \dot{-} \psi$

($\dot{-}$7) $K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} (\phi \wedge \psi)$

($\dot{-}$8) If $\phi \notin K \dot{-} (\phi \wedge \psi)$, then $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \phi$

The reader familiar with the AGM postulates for contraction will notice that postulate ($\dot{-}$3) is given in a slightly weaker form than usual [8, p. 61]. The usual consequent, $K \dot{-} \phi = K$, is easily recovered with the help of ($\dot{-}$2). It also follows from ($\dot{-}$1), ($\dot{-}$2) and ($\dot{-}$5) that $K \dot{-} \phi = K$ for every $\phi \in Cn\,(\emptyset)$. This condition is sometimes referred to as *Failure* (c.f. [14, p. 109]).

The most controversial of the AGM postulates for contraction is ($\dot{-}$5) which is commonly referred to as *Recovery*. In the presence of postulates ($\dot{-}$1) and ($\dot{-}$2) it implies that $K = Cn\,((K \dot{-} \phi) \cup \{\phi\})$ if ϕ is in K. That is, removing a sentence ϕ and then restoring it leads to the original belief set whenever ϕ is in that belief set to begin with. Interestingly, recovery has no counterpart among the postulates for AGM revision[14] which may be defined from contraction via the Levi Identity.[15] We shall not enter into the polemic surrounding the recovery property but, instead, refer the interested reader to the relevant literature [13, 18, 22, 23, 25].

We find it important to also consider the following weaker versions of ($\dot{-}$7) and ($\dot{-}$8). We note, given postulates ($\dot{-}$1) — ($\dot{-}$6), that ($\dot{-}$7) implies ($\dot{-}$7c) and ($\dot{-}$8) implies ($\dot{-}$8c) (see [30, Lemma 1]).

($\dot{-}$7c) If $\psi \in K \dot{-} (\phi \wedge \psi)$, then $K \dot{-} \phi \subseteq K \dot{-} (\phi \wedge \psi)$

($\dot{-}$8c) If $\psi \in K \dot{-} (\phi \wedge \psi)$, then $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \phi$

The derivation of ($\dot{-}$7c) from ($\dot{-}$7) uses ($\dot{-}$5), while the derivation of ($\dot{-}$8c) from ($\dot{-}$8) uses ($\dot{-}$4) and Failure. Postulates ($\dot{-}$7) and ($\dot{-}$8) are, respectively, the contraction counterparts of the rules Or and Rational Monotony used in nonmonotonic reasoning. On the other hand, postulates ($\dot{-}$7c) and ($\dot{-}$8c) are the contraction counterparts of the rules Cut and Cumulative Monotony respectively.[16] In nonmonotonic reasoning, Cut and Cumulative Monotony are considered to be much more fundamental than Or and Rational Monotony. Postulates ($\dot{-}$7c) and ($\dot{-}$8c) are indeed exceedingly plausible in the context of belief revision as well. Taken together, they state that if ψ is still present after the removal of $\phi \wedge \psi$, then that removal just boils down to the removal of ϕ.

6. Postulates for Severe Withdrawals

As we shall soon see, the following postulates characterise the new belief removal operation advocated in sections 3 and 4. The most obvious difference with AGM contractions is marked by the absence of the Recovery postulate.[17]

($\ddot{-}$1) $\quad K \ddot{-} \phi = Cn\,(K \ddot{-} \phi)$

($\ddot{-}$2) $\quad K \ddot{-} \phi \subseteq K$

($\ddot{-}$3) $\quad$ If $\phi \notin K$ or $\vdash \phi$, then $K \subseteq K \ddot{-} \phi$

($\ddot{-}$4) $\quad$ If $\nvdash \phi$, then $\phi \notin K \ddot{-} \phi$

($\ddot{-}$6) $\quad$ If $Cn\,(\phi) = Cn\,(\psi)$, then $K \ddot{-} \phi = K \ddot{-} \psi$

($\ddot{-}$7a) $\quad$ If $\nvdash \phi$, then $K \ddot{-} \phi \subseteq K \ddot{-} (\phi \wedge \psi)$

($\ddot{-}$8) $\quad$ If $\phi \notin K \ddot{-} (\phi \wedge \psi)$, then $K \ddot{-} (\phi \wedge \psi) \subseteq K \ddot{-} \phi$

Postulates ($\ddot{-}$1), ($\ddot{-}$2) ($\ddot{-}$4), ($\ddot{-}$6) and ($\ddot{-}$8) are simply those for AGM contraction over K. Postulate ($\ddot{-}$3) contains an additional antecedent in order to take care of the limiting case of Failure (which was previously handled with the aid of Recovery). We shall call the collection ($\ddot{-}$1), ($\ddot{-}$2), ($\ddot{-}$3), ($\ddot{-}$4) and ($\ddot{-}$6) the *basic* postulates. Postulate ($\ddot{-}$7) has been replaced by the much stronger *antitony condition* ($\ddot{-}$7a). It states that anything that is given up in order to remove a strong sentence (the conjunction of ϕ and ψ) should also be given up when removing a weaker sentence (ϕ) from the belief set, provided the latter is not logically true. Intuitively, this makes quite a bit of sense. In giving up $\phi \wedge \psi$ at least one of ϕ or ψ must be abandoned. If ϕ is given up in $K \ddot{-} (\phi \wedge \psi)$, we can simply achieve $K \ddot{-} \phi$ by abandoning the same beliefs. If ψ is given up instead, we may have to give up more. If we are serious about adhering to the Principles of Preference and Indifference we should at least give up as much because ψ and the beliefs that have been given up thus far were apparently held in lower regard.

Clearly, the postulate of recovery does not follow from the present collection of postulates. Makinson [23] refers to belief removal operations satisfying postulates ($\dot{-}$1) – ($\dot{-}$4) and ($\dot{-}$6), but not necessarily Recovery, as *withdrawal functions*. In this sense, any withdrawal function would be considered weaker than an AGM contraction function. On the other hand, we have decisively strengthened ($\dot{-}$7) through its replacement by ($\ddot{-}$7a). In this respect (and by the introduction of the Failure condition in ($\ddot{-}$3)), the resulting withdrawal function is stronger than an AGM contraction function. For reasons that will become clear later, we call the operations characterised by the above set of postulates *severe withdrawal functions*.

Notice also, in the context of the basic postulates ($\ddot{-}$1) – ($\ddot{-}$4) and ($\ddot{-}$6), that ($\ddot{-}$7a) implies ($\ddot{-}$7c) and ($\ddot{-}$8) implies ($\ddot{-}$8c). Recovery is not required for these derivations.

An alternative axiomatisation of severe withdrawal is given by Pagnucco [27]. It consists of the AGM postulates ($\ddot{-}$1) – ($\ddot{-}$4) and ($\ddot{-}$6) together with the following two postulates:

($\ddot{-}$9) If $\phi \notin K \ddot{-} \psi$, then $K \ddot{-} \psi \subseteq K \ddot{-} \phi$

($\ddot{-}$10) If $\not\vdash \phi$ and $\phi \in K \ddot{-} \psi$, then $K \ddot{-} \phi \subseteq K \ddot{-} \psi$

It is shown that postulates ($\ddot{-}$7) and ($\ddot{-}$8) follow from these postulates. In fact, postulate ($\ddot{-}$10) is redundant as we show below. We shall soon see (Lemma 3) that these postulates do not hold in general for AGM contraction.

Postulate ($\ddot{-}$9) states that, if ϕ is given up in removing ψ from K, then anything given up in removing ϕ from K should also be given up when removing ψ from K. Casting our thoughts back to the principles outlined at the outset, the antecedent tells us that ϕ is held in no higher regard than ψ (possibly lower) and therefore no more (perhaps less) need be given up in order to remove ψ when compared to removing ϕ. That is, at least as much work needs to be done in removing ψ as is required to remove ϕ from K. Postulate ($\ddot{-}$10) states that, if a non-tautological sentence ϕ is retained when removing ψ from K, then whatever is given up to remove ψ should also be given up to remove ϕ from K. When ϕ is held in higher regard than ψ, more work may need to be done in giving up ϕ than is required to give up ψ.

The following lemma shows that these two proposed axiomatisations are equivalent.

LEMMA 1. *Let the basic postulates ($\ddot{-}$1) – ($\ddot{-}$4) and ($\ddot{-}$6) be given. Then*
(i) ($\ddot{-}$7a) and ($\ddot{-}$8) taken together are equivalent with ($\ddot{-}$9);
(ii) ($\ddot{-}$7a) and ($\ddot{-}$8) imply ($\ddot{-}$10).

The second part shows that postulate ($\ddot{-}$10) is indeed redundant and we can omit it from further consideration although it is of course a property of severe withdrawals. Let us briefly look at some further properties following from our postulates in order to gain a clearer insight into the nature of severe withdrawal.

LEMMA 2. *Let* $\ddot{-}$ *be a severe withdrawal function over* K. *Then*

(i) *Either* $K \ddot{-} \phi \subseteq K \ddot{-} \psi$ *or* $K \ddot{-} \psi \subseteq K \ddot{-} \phi$.

(ii) *Either* $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \phi$ *or* $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \psi$.

(iii) *If* $K \ddot{-} \phi \wedge \psi \subseteq K \ddot{-} \psi$, *then* $\psi \notin K \ddot{-} \phi$ *or* $\vdash \phi$ *or* $\vdash \psi$.

(iv) *If* $\nvdash \phi$ *and* $\nvdash \psi$, *then either* $\phi \notin K \ddot{-} \psi$ *or* $\psi \notin K \ddot{-} \phi$.

The first part of the lemma tells us that severe withdrawals are nested one within the other. This attests to the strength of the introduced postulates. The second part states that withdrawal by a conjunction is equivalent to withdrawal by of one its conjuncts (give up the least preferred). This factoring condition, called *Decomposition* [1, p. 525], characterises maxichoice contraction within the class of AGM partial meet contraction functions [1, Observation 6.3(a)]. In the current context however, we are concerned with withdrawal functions and thus recovery is lacking. The third part of the lemma is the condition called *Converse Conjunctive Inclusion* in Fermé and Rodriguez [7, p. 4]. Our proof shows that this condition is redundant in the axiomatisation of these authors (which includes ($\ddot{-}$9)). The last property is referred to as *Expulsiveness* [15, Observation 2.52].[18] It says that for any two arbitrary non-theorems ϕ and ψ, in the removal of one of them the other will also be removed. Expulsiveness is an undesirable property since we do not necessarily want sentences that intuitively have nothing to do with one another to affect each other in belief contractions. This is the bitter pill we have to swallow if we want to adhere to the Principles of Indifference and Preference.

Before we progress it will be useful to adopt some uniform terminology in order to better classify the belief removal operations we have come across thus far. This will also serve to give a clearer picture of how severe withdrawal fits into the overall scheme of such functions.

DEFINITION 1. Any function $\dot{-}$ satisfying ($\dot{-}$1) – ($\dot{-}$4), ($\dot{-}$6) is referred to as a *withdrawal function*. Moreover, any function $\dot{-}$ satisfying ($\dot{-}$1) – ($\dot{-}$4), ($\dot{-}$6) and

$$\left\{\begin{array}{c} (\dot{-}7c), (\dot{-}8c) \\ (\dot{-}7), (\dot{-}8c) \\ (\dot{-}7), (\dot{-}8) \\ (\ddot{-}3), (\ddot{-}7a) \textit{ and } (\dot{-}8) \end{array}\right\} \text{ is called a } \left\{\begin{array}{c} \textit{cumulative withdrawal} \\ \textit{preferential withdrawal} \\ \textit{rational (or AGM) withdrawal} \\ \textit{severe withdrawal} \end{array}\right\}$$

function.

Any withdrawal function satisfying the Recovery postulate ($\dot{-}$5) is called a *contraction* function.

It should be clear from the foregoing discussion that the class of withdrawals (without recovery) form a linear hierarchy. The labels ‘cumulative’, 'preferential' and 'rational' are borrowed from the corresponding notions in the theory of nonmonotonic reasoning [24, 30]. Severe withdrawal is the most restricted class of those present and then, in order of increasing generality, we have rational (or, AGM) withdrawals, preferential withdrawals, cumulative withdrawals and (unrestricted) withdrawals. Adding recovery to a withdrawal function leads to the corresponding contraction function. What, then, is the nature of the even more restricted class of *severe contraction functions*? Significantly, no such contraction function exists (on pain of triviality).

LEMMA 3. *There is no contraction function over a non-trivial belief set K that satisfies postulates $(\dot{-}1)$ – $(\dot{-}8)$ and $(\ddot{-}9)$.*

Here we call a belief set K *trivial* if it does not contain a non-tautological sentence ϕ that does not already axiomatise K, i.e., for which $K \neq Cn(\phi)$.

We are thus faced with genuine alternatives. It is clear that $(\ddot{-}7\mathrm{a})$ is not satisfied by AGM contractions. Unable to obtain a hybrid of AGM contraction and severe withdrawal however, we devote the remainder of this paper to explaining the relationship between the two and providing two precise AGM-like constructive modellings for severe withdrawal functions.

7. Relating AGM Contraction and Severe Withdrawals

Thus far we have motivated our investigation of belief removal primarily through the intuition behind two constructive modellings dealt with in Sections 3 and 4 and the way in which they satisfy certain principles of rationality. However, we can study the correspondence between AGM contractions and severe withdrawals without reference to systems of spheres or entrenchment relations and, in fact, without reference to any constructive modelling at all. Severe withdrawals are far more "skeptical" than AGM contractions in that they lead to theories that are smaller in terms of set-theoretic inclusion. This is but one of the interesting relationships between the two.

Makinson [23] observes that withdrawal functions can be partitioned into revision equivalent classes. Two withdrawal functions, $\dot{-}$ and $\ddot{-}$ say, are *revision equivalent* if the corresponding revision functions, defined from them via the Levi Identity, are equivalent, i.e., if $K{*}\phi = Cn((K \dot{-} \neg\phi) \cup \{\phi\}) = Cn((K \ddot{-} \neg\phi) \cup \{\phi\}) = K \ddot{*} \phi$ for all ϕ in $\mathcal{L}$. Moreover, he noted [23, Observation p. 389] that in each revision equivalent class $[\dot{-}]$, the maximal element (in terms of set-theoretic inclusion) was an AGM (partial meet) contraction function.

The problem addressed in this section is to find the correspondence between revision equivalent AGM contraction functions and severe withdrawal functions. The idea at the back of our minds is that the relationship between matching functions should be exactly as that in the constructions by means of systems of spheres or entrenchment relations. When talking about *the* correspondence we presuppose that there is a unique AGM contraction function and a unique severe withdrawal function in each class $[\dot{-}]$ of revision equivalent withdrawal functions. The following lemma shows that this is indeed the case.

LEMMA 4. *Let* $\dot{-}$ *and* $\dot{-}'$ *be two withdrawal functions that are revision equivalent. Then* $\dot{-}$ *and* $\dot{-}'$ *are identical whenever either of the following two clauses holds:*

(i) $\dot{-}$ *and* $\dot{-}'$ *satisfy* $(\dot{-}1)$, $(\dot{-}2)$ *and Recovery* $(\dot{-}5)$*;*

(ii) $\dot{-}$ *and* $\dot{-}'$ *are severe withdrawal functions.*

Building on earlier results of Gärdenfors, Makinson [23, p. 389] gives a somewhat roundabout proof of the fact that there is only one element in $[\dot{-}]$ which satisfies $(\dot{-}1)$ – $(\dot{-}6)$. Part *(i)* of the above lemma shows that Recovery almost alone guarantees an identity in this case. On the other hand, lacking Recovery, the proof for severe withdrawals in part *(ii)* makes essential use of postulates $(\dot{-}7a)$ and $(\dot{-}8c)$.

The constructive modellings considered in Sections 3 and 4 have indicated that, for a severe withdrawal function $\ddot{-}$ and its revision equivalent AGM contraction function $\dot{-}$, the belief set $K\ddot{-}\phi$ will contain no more beliefs than $K\dot{-}\phi$.[19] Letting $\dot{-}$ be an AGM contraction function, the corresponding severe withdrawal function $\ddot{-}$ can be defined as follows.

(Def $\ddot{-}$ from $\dot{-}$) $\quad K\ddot{-}\phi = \begin{cases} \{\psi : \psi \in K\dot{-}(\phi \wedge \psi)\} & \text{if } \nvdash \phi \\ K & \text{otherwise} \end{cases}$

Intuitively, in giving up ϕ, (Def $\ddot{-}$ from $\dot{-}$) tells us to retain those beliefs ψ that would be retained when given a choice to remove either ϕ or ψ (or both). If ψ is considered more important than ϕ when there is a possibility of deciding between them, then this consideration should also be kept in mind when deciding what to remove in the severe withdrawal of K by ϕ.

An alternative idea is expressed by the following definition.

(Def′ $\ddot{-}$ from $\dot{-}$) $\quad K\ddot{-}\phi = \begin{cases} \bigcap\{K\dot{-}(\phi \wedge \psi) : \psi \in \mathcal{L}\} & \text{if } \nvdash \phi \\ K & \text{otherwise} \end{cases}$

According to (Def′ $\ddot{-}$ from $\dot{-}$), in giving up ϕ we should retain those beliefs that are always retained when given a choice between giving up ϕ or another belief. That is, we retain those beliefs that are always retained when there is the possibility of removing either ϕ or another sentence (or both).

It turns out that these two approaches are, in fact, equivalent.

LEMMA 5. *If* $\dot{-}$ *satisfies* $(\dot{-}1)$, $(\dot{-}2)$, $(\dot{-}5)$, $(\dot{-}6)$, $(\dot{-}7)$ *and* $(\dot{-}8)$, *then* (Def $\ddot{-}$ from $\dot{-}$) *and* (Def′ $\ddot{-}$ from $\dot{-}$) *are equivalent.*

It remains, however, to show that these definitions are in fact adequate. Moreover, we ought to show that revision equivalent severe withdrawals are (set-theoretically) smaller than AGM contractions. The relevant result is as follows. result.

OBSERVATION 6. *If $\dot{-}$ is an AGM contraction function, then $\ddot{-}$ as obtained by (Def $\ddot{-}$ from $\dot{-}$) is a severe withdrawal function revision equivalent to $\dot{-}$, and $K\ddot{-}\phi \subseteq K\dot{-}\phi$ for all $\phi \in \mathcal{L}$.*

To indicate that severe withdrawals are in fact very severe compared to other withdrawals in regard to the volume of beliefs removed, we note the following result.

OBSERVATION 7. *Let $\dot{-}$ be an AGM contraction function. Then the severe withdrawal function $\ddot{-}$ defined from $\dot{-}$ by definition (Def $\ddot{-}$ from $\dot{-}$) is the smallest withdrawal function satisfying postulate ($\dot{-}$8c) which is revision equivalent to $\dot{-}$.*

Smallness is measured here in terms of set-theoretic inclusion. Thus, severe withdrawal removes more beliefs than a large class of (revision equivalent) withdrawals which encompasses cumulative, preferential and rational withdrawals (as well as their contraction counterparts of course). This is an interesting and significant class of belief removal functions because they satisfy the contraction counterpart ($\dot{-}$8c) of Cumulative Monotony which is an important and widely accepted property in the study of (nonmonotonic)consequence relations.

However, severe withdrawals are not the smallest withdrawal functions. This distinction belongs to a more iron-fisted or procrustean withdrawal function which may be defined as follows.

(Def $\dddot{-}$ from $\dot{-}$) $\quad K\dddot{-}\phi = \begin{cases} Cn(\phi) \cap K\dot{-}\phi & \text{if } \phi \in K \text{ and } \nvdash \phi \\ K & \text{otherwise} \end{cases}$

This definition would work equally well with $\ddot{-}$ substituted for $\dot{-}$. It determines an excessive type of belief removal. First we show that it is, in fact, the smallest revision equivalent withdrawal function.

OBSERVATION 8. *Let $\dot{-}$ be an AGM contraction function. Then the withdrawal function $\dddot{-}$ defined from $\dot{-}$ by definition (Def $\dddot{-}$ from $\dot{-}$) is the smallest withdrawal function which is revision equivalent to $\dot{-}$.*

From (Def $\dddot{-}$ from $\dot{-}$) it is easy to see that the following property holds: *If $\phi \in K$ and $\nvdash \phi$, then $K\dddot{-}\phi \subseteq Cn(\phi)$.* Such a withdrawal, uniformly applied, is drastic indeed and it is also counterintuitive. Why should we retain only consequences of the very belief we want to retract?

Returning to our discussion of the relationship between severe withdrawal and AGM contraction, going back in the other direction (i.e., from $\ddot{-}$ to $\dot{-}$) is quite simple. Let $\ddot{-}$ be a severe withdrawal function. Then the corresponding AGM contraction function $\dot{-}$ is defined by

(Def $\dot{-}$ from $\ddot{-}$) $\quad K \dot{-} \phi = \begin{cases} K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\}) & \text{if } \nvdash \phi \\ K & \text{otherwise} \end{cases}$

This method consists of consecutively applying the Levi and Harper identities. It has been advocated as a trick of enforcing the Recovery postulate by Makinson [23, pp. 389, 391]. Its adequacy is demonstrated by the following result.

OBSERVATION 9. *If $\ddot{-}$ is a severe withdrawal function, then $\dot{-}$ as obtained by (Def $\dot{-}$ from $\ddot{-}$) is an AGM contraction function revision equivalent to $\ddot{-}$, and $K \ddot{-} \phi \subseteq K \dot{-} \phi$ for all $\phi \in \mathcal{L}$.*

The appropriateness of the definitions in this section is further indicated by the following result demonstrating that $\dot{-}$ and $\ddot{-}$ induce isomorphic structures [5] via the definitions above. The first part states that successive applications of (Def $\ddot{-}$ from $\dot{-}$) and (Def $\dot{-}$ from $\ddot{-}$), in that order, result in the same AGM contraction function. The second part states that the corresponding result, mutatis mutandis, holds for severe withdrawal functions.

OBSERVATION 10. *(i) If we start with an AGM contraction function $\dot{-}$, turn it into a severe withdrawal function $\ddot{-}$ by (Def $\ddot{-}$ from $\dot{-}$) and turn the latter into an AGM contraction function $\dot{-}'$ by (Def $\dot{-}$ from $\ddot{-}$), then we end up with $\dot{-}' = \dot{-}$.*
(ii) If we start with a severe withdrawal function $\ddot{-}$, turn it into an AGM contraction function $\dot{-}$ by (Def $\dot{-}$ from $\ddot{-}$) and turn the latter into a severe withdrawal function $\ddot{-}'$ by (Def $\ddot{-}$ from $\dot{-}$), then we end up with $\ddot{-}' = \ddot{-}$.

This result implies that (Def $\ddot{-}$ from $\dot{-}$) and (Def $\dot{-}$ from $\ddot{-}$) induce a one-one correspondence between (revision equivalent) AGM contraction functions and severe withdrawal functions.

In this section we have taken a closer look at the interrelationship between AGM contraction and severe withdrawal. One very important point to notice is that, although we are contrasting different belief removal behaviour, there is no effect on the respective revision operations obtained via the Levi Identity. Since different revision behaviour is not, in general, linked with identical belief removal operations, there is a greater degree of freedom in belief removal than in belief revision functions. It is our aim here to indicate that there are types of belief removal behaviour, differing from AGM contraction yet revision equivalent to it, that can be motivated by rational means. In fact, our main aim is to promote severe withdrawal as a highly principled member of this community. We have also witnessed another member — viz. the “iron-fisted” withdrawal — which is the smallest withdrawal function in a class of revision equivalent withdrawals. Makinson [23, p. 389] points out that AGM contraction is the largest withdrawal function in this class. Other possibilities to be found in the literature include Levi's [18] saturatable contractions (using undamped informational value) and the partial meet variety studied by Hansson and Olsson [14], Levi [18, 20] contraction using damped informational value of type 1, Levi [19] contraction using damped informational value of type 2 (i.e., *mild contractions* or, using our terminology, *severe withdrawal*), Cantwell's [4] fallback-based contraction, Meyer *et al.*'s systematic withdrawal [26], Lindström and Rabinowicz's [22] interpolation operator, Fermé and Rodriguez's [6] semi-contraction operator and Nayak's [p.c.] withdrawal. Appendix B briefly contrasts these various approaches in terms of systems of spheres.

Having investigated the relationship between severe withdrawal and AGM contraction functions, we now return to the system of spheres construction for belief removal functions.

8. Retrieving Systems of Spheres from Rational Withdrawals

In this section we elaborate upon the ideas presented in Section 3 in a more technical manner. Grove [12] views maximally consistent sets of sentences (consistent complete theories) as "possible worlds". An ordering is then imposed over the set of all such possible worlds $\mathcal{M}_{\mathcal{L}}$. The set of all possible worlds consistent with a set of sentences K (not necessarily closed under Cn) is denoted $[K]$ and may be determined as $[K] = \{m \in \mathcal{M}_{\mathcal{L}} : K \subseteq m\}$. We use $[\phi]$ as a shorthand for $[\{\phi\}]$. We also define a function $th : 2^{\mathcal{M}_{\mathcal{L}}} \to \mathcal{K}$ mapping sets of possible worlds to belief sets by putting $th(X) = \bigcap X$ for any $X \subseteq \mathcal{M}_{\mathcal{L}}$.

Now recall that a system of spheres is a nested collection of sets of worlds in which $[K]$ is the smallest sphere and $\mathcal{M}_{\mathcal{L}}$ is the largest. Formally, we have the following definition due to Adam Grove.

DEFINITION 2. [12] Let $\mathcal{S}$ be any collection of subsets of $\mathcal{M}_{\mathcal{L}}$. We call $\mathcal{S}$ a system of spheres, centred on $X \subseteq \mathcal{M}_{\mathcal{L}}$, if it satisfies the following conditions:

($\mathcal{S}1$) $\mathcal{S}$ is totally ordered by $\subseteq$; that is, if $U,\ V \in \mathcal{S}$, then $U \subseteq V$ or $V \subseteq U$

($\mathcal{S}2$) X is the $\subseteq$-minimum of $\mathcal{S}$

($\mathcal{S}3$) $\mathcal{M}_{\mathcal{L}}$ is the $\subseteq$-maximum of $\mathcal{S}$

($\mathcal{S}4$) If $\phi \in \mathcal{L}$ and $\nvdash \neg\phi$, then there is a smallest sphere in $\mathcal{S}$ intersecting $[\phi]$ (i.e., there is a sphere $U \in \mathcal{S}$ such that $U \cap [\phi] \neq \emptyset$, and $V \cap [\phi] \neq \emptyset$ implies $U \subseteq V$ for all $V \in \mathcal{S}$)

That spheres are nested is specified by condition ($\mathcal{S}1$). Condition ($\mathcal{S}4$) guarantees that there is a smallest or innermost sphere intersecting $[\phi]$ for any $\phi \in \mathcal{L}$. This corresponds to Lewis' [21, p. 19] limit assumption. We denote this sphere $c_{\mathcal{S}}(\phi)$ (cf. Section 3). More formally we have a function $c_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$ defined as follows:

$$\text{(Def } c_{\mathcal{S}}\text{)} \qquad c_{\mathcal{S}}(\phi) = \begin{cases} \text{the sphere } U \in \mathcal{S} \text{ such that} & \\ \qquad U \cap [\phi] \neq \emptyset \text{ and } V \cap [\phi] \neq \emptyset & \\ \qquad \text{implies } U \subseteq V \text{ for all } V \in \mathcal{S} & \text{whenever } \nvdash \neg\phi \\ [K] & \text{otherwise} \end{cases}$$

This allows us to formally define a function $f_{\mathcal{S}} : \mathcal{L} \to 2^{\mathcal{M}_{\mathcal{L}}}$ returning the ϕ-worlds closest to $[K]$ (cf. Section 3). With each system of spheres $\mathcal{S}$ centred on $[K]$ we can associate a function $f_{\mathcal{S}}(\phi) = [\phi] \cap c_{\mathcal{S}}(\phi)$. Note that, in the case where $\vdash \phi$, by (Def $c_{\mathcal{S}}$), we automatically have via (Def $\dot{-}$ from $\mathcal{S}$) and (Def $\ddot{-}$ from $\mathcal{S}$) that $K \dot{-} \phi = K \ddot{-} \phi = K$.

Our main interest in this section is the method used to construct the system of spheres centred on $[K]$ corresponding to an AGM contraction or severe withdrawal function. Before turning to severe withdrawal functions, we first adapt Lewis' [21, pp. 59 and 133–134] and Grove's [12, p. 162] methods of constructing systems of spheres from counterfactuals and revisions, respectively, to the context of AGM contraction functions. The idea is to specify a method by which each sphere X_ϕ (the minimal sphere intersecting $[\neg\phi]$) can be determined. A system of spheres $\mathcal{S}$ is then obtained by accumulating all sets X_ϕ so determined and the set $\mathcal{M}_{\mathcal{L}}$ of all worlds just in case it is not identified with one of the X_ϕ's. More specifically, $\mathcal{S} = \{X_\phi : \phi \in \mathcal{L}\} \cup \{\mathcal{M}_{\mathcal{L}}\}$ whenever $K \neq \mathcal{L}$ and $\mathcal{S} = \{X_\phi : \phi \in \mathcal{L}\} \cup \{\mathcal{M}_{\mathcal{L}}\} \cup \{\emptyset\}$ otherwise.

A set of possible worlds $X \subseteq \mathcal{M}_{\mathcal{L}}$ is in the Lewis-Grovean system of spheres $\mathcal{S}$ (i.e., is a sphere in $\mathcal{S}$) derived from $\dot{-}$ if and only if

$$X \subseteq \{m : \text{there is a } \phi \text{ such that } K \dot{-} \phi \subseteq m\} \text{ and}$$

$$\text{for all } \psi, \text{ if } X \cap [\neg\psi] \neq \emptyset \text{ then } [K \dot{-} \psi] \subseteq X.$$

This condition, which we shall refer to as the *first* construction of Lewis and Grove, can be rephrased by the following equation:

$$X \text{ is in } \mathcal{S} \text{ if and only if } X = \bigcup \{[K \dot{-} \psi] : X \not\subseteq [\psi]\}^{20}.$$

However, this is not what is actually used in the completeness proofs of Lewis and Grove. The spheres X_ϕ they need for their proofs (Lewis [21, p. 59], Grove [12, p. 162]) have the following form, here again transferred from the context of counterfactuals and revisions to the context of contractions. A set X_ϕ of worlds is in $\mathcal{S}$ obtained from $\dot{-}$ if and only if

$$\text{(Def } \mathcal{S} \text{ from } \dot{-}\text{)} \qquad X_\phi = \begin{cases} \bigcup\{[K \dot{-} \psi] : [\psi] \subseteq [\phi]\} & \text{whenever } \nvdash \phi \\ [K] & \text{otherwise} \end{cases}$$

We shall refer to this condition as the *second* construction of Lewis and Grove. Every X_ϕ thus constructed is a Lewis-Grovean sphere according to the first construction and it is actually the $\subseteq$-*minimal* such sphere intersecting $[\neg\phi]$.[21] However, there is no guarantee that all spheres of the first construction can be captured by the X_ϕ's. Using either of the first or the second Lewis-Grove construction results in a system of spheres where each sphere can be represented as the union of model sets of a certain collection of theories.

The situation changes if we consider severe withdrawal instead of AGM contraction. If the second construction is applied to severe withdrawals, then we shall see that each sphere consists of the model set of exactly one theory.[22]

In order to construct a system of spheres $\mathcal{S}$ centred on $[K]$ from a severe withdrawal function $\ddot{-}$ over K, we essentially identify a sphere in $\mathcal{S}$ with the collection $[K \ddot{-} \phi]$ for some $\phi \in \mathcal{L}$. Any worlds not accounted for in this manner (i.e., "irrelevant worlds" — see below) are thrown into the outermost sphere $\mathcal{M}_\mathcal{L}$ (by the construction of $\mathcal{S}$ noted above). More precisely, we have:

(Def $\mathcal{S}$ from $\ddot{-}$) $\quad X_\phi = [K \ddot{-} \phi]$

Note that we do not need a special case for $\not\vdash \phi$ because in this scenario, due to the Failure property captured by postulates ($\ddot{-}$2) and ($\ddot{-}$3), we have $K \ddot{-} \phi = K$ so $X_\phi = [K]$. We now show that for severe withdrawal functions $\ddot{-}$, this definition coincides with the second Lewis-Grove condition (Def $\mathcal{S}$ from $\dot{-}$).

LEMMA 11. *If* $\ddot{-}$ *is a severe withdrawal function, then the two conditions* (Def $\mathcal{S}$ from $\dot{-}$) *and* (Def $\mathcal{S}$ from $\ddot{-}$) *are equivalent.*

This result also highlights the special nature of severe withdrawal functions. Due to their properties, we obtain a much simplified way to construct systems of spheres.

We now briefly investigate several transformations that may be applied to a system of spheres without affecting the AGM contraction or severe withdrawal generated from it. They give rise to systems of spheres that are equivalent in the sense of the following definition.

DEFINITION 3. Let $\mathcal{S}$ and $\mathcal{S}'$ be two systems of spheres, let $\dot{-}$ and $\dot{-}'$ be the contraction functions based on $\mathcal{S}$ and $\mathcal{S}'$ and $\ddot{-}$ and $\ddot{-}'$ be the severe withdrawal functions based on $\mathcal{S}$ and $\mathcal{S}'$, respectively. Then $\mathcal{S}$ and $\mathcal{S}'$ are called *equivalent* if and only if for every sentence ϕ it holds that $K \dot{-} \phi = K \dot{-}' \phi$ and $K \ddot{-} \phi = K \ddot{-}' \phi$.

Consider, now, the following operations on systems of spheres.

DEFINITION 4. Let $\mathcal{S}$ be a system of spheres centred on $[K]$. Then

$\mathcal{S}_t$ (the *trimming* of $\mathcal{S}$) is obtained by removing from $\mathcal{S}$ all the spheres S such that S is never the smallest sphere intersecting $[\phi]$, i.e., all S such that $S \neq c_{\mathcal{S}}(\phi)$ for all $\phi \in \mathcal{L}$.

$\mathcal{S}_u$ (the *closure under unions* of $\mathcal{S}$) is obtained by adding to $\mathcal{S}$ all the unions U of classes of spheres in $\mathcal{S}$, i.e., all U such that $U = \bigcup \mathcal{S}'$ for some subset $\mathcal{S}'$ of $\mathcal{S}$.

$\mathcal{S}_{cl}$ (the *topological closure* of $\mathcal{S}$) is obtained by replacing all spheres S in $\mathcal{S}$ by the sets of worlds (models) that satisfy the theory of S, i.e., by replacing all S in $\mathcal{S}$ by $[th(S)]$.

In this last case we can define an operator cl on sets of worlds (models) as $cl(S) = [th(S)]$. This operation is clearly a closure operator, i.e., *(i)* $S \subseteq cl(S)$, *(ii)* $cl(cl(S)) \subseteq cl(S)$ and *(iii)* $S \subseteq S'$ implies $cl(S) \subseteq cl(S')$. Moreover, S and $cl(S)$ have the same theory, $th(S) = th(cl(S))$. Notice that for every $\phi \in \mathcal{L}$, $th(c_{\mathcal{S}}(\phi)) = th(c_{\mathcal{S}_{cl}}(\phi))$ and $th(f_{\mathcal{S}}(\phi)) = th(f_{\mathcal{S}_{cl}}(\phi))$. Clearly, all the operations on $\mathcal{S}$ result in systems of spheres, and $\mathcal{S}_t \subseteq \mathcal{S} \subseteq \mathcal{S}_u$, but $\mathcal{S}_{cl}$ is in general not comparable to any of the other systems of spheres. Nevertheless, we have the following result relating systems of spheres and transformations applied to them.

LEMMA 12. *$\mathcal{S}$, $\mathcal{S}_t$, $\mathcal{S}_u$ and $\mathcal{S}_{cl}$ are all equivalent.*

The final result in this section lends further weight to the suitability of the pairing of AGM contractions and severe withdrawals that we suggested in Section 7. It shows that any two functions related by the appropriate definitions generate equivalent systems of spheres.

OBSERVATION 13. *Let $\dot{-}$ and $\ddot{-}$ be corresponding AGM contraction and severe withdrawal functions either via (Def $\ddot{-}$ from $\dot{-}$) or via (Def $\dot{-}$ from $\ddot{-}$). Then $\dot{-}$ and $\ddot{-}$ lead to equivalent systems of spheres, via (Def $\mathcal{S}$ from $\dot{-}$) and (Def $\mathcal{S}$ from $\ddot{-}$). More precisely, the system of spheres obtained from $\ddot{-}$ is the topological closure of that obtained from $\dot{-}$.*

For a contraction or withdrawal function $\dot{-}$, call a world m *irrelevant* (with respect to $\dot{-}$) if there is no $\phi \in \mathcal{L}$ such that m contains $K \dot{-} \phi$ (i.e., $m \notin [K \dot{-} \phi]$). Lewis-Grove spheres relegate irrelevant worlds to the outermost sphere, thereby sacrificing the Δ-elementarity of their spheres. In severe withdrawal we, as it were, add the irrelevant worlds to the Lewis-Grove spheres in such a way that they "make no difference" for withdrawals but make sure that the resulting spheres are Δ-elementary.[23]

9. Representation Theorems for Sphere-Based Withdrawals

The following analogues of Grove's results concerning belief contraction (which we state without proof) show that the construction in terms of systems of spheres outlined in Sections 3 and 8 is in fact an appropriate rendering of the AGM rationality postulates for belief contraction over K.[24] The first part of the observation states that the method of adding the innermost $\neg\phi$-worlds to $[K]$ as described by (Def $\dot{-}$ from $\mathcal{S}$) does indeed produce an AGM contraction function. The second part shows that for any AGM contraction function and belief set K, one can construct a system of spheres $\mathcal{S}$ centred on $[K]$ using (Def $\mathcal{S}$ from $\dot{-}$) for which the addition of the innermost $\neg\phi$-worlds to $[K]$ corresponds to the contraction of K by ϕ.

OBSERVATION 14. [12, Theorems 1 and 2] *(i) If $\mathcal{S}$ satisfies ($\mathcal{S}1$) – ($\mathcal{S}4$), then the function $\dot{-}$ obtained from $\mathcal{S}$ by* (Def $\dot{-}$ from $\mathcal{S}$) *is an AGM contraction function.*
(ii) If $\dot{-}$ is an AGM contraction function, then $\dot{-}$ can be represented as a sphere-based contraction, where the sphere system $\mathcal{S}$ on which $\dot{-}$ is based is obtained by (Def $\mathcal{S}$ from $\dot{-}$) *and $\mathcal{S}$ satisfies ($\mathcal{S}1$) – ($\mathcal{S}4$).*

This result shows the mutual adequacy of definitions (Def $\dot{-}$ from $\mathcal{S}$) and (Def $\mathcal{S}$ from $\dot{-}$) introduced here. The corresponding representation theorem can now be established for severe withdrawal over K.

OBSERVATION 15. *(i) If $\mathcal{S}$ satisfies ($\mathcal{S}1$) – ($\mathcal{S}4$), then the function $\ddot{-}$ obtained from $\mathcal{S}$ by* (Def $\ddot{-}$ from $\mathcal{S}$) *is a severe withdrawal function.*
(ii) If $\ddot{-}$ is a severe withdrawal function, then $\ddot{-}$ can be represented as a sphere-based withdrawal, where the sphere system $\mathcal{S}$ on which $\ddot{-}$ is based is obtained by (Def $\mathcal{S}$ from $\dot{-}$) *(or equivalently, by* (Def $\mathcal{S}$ from $\ddot{-}$)*) and $\mathcal{S}$ satisfies ($\mathcal{S}1$) – ($\mathcal{S}4$).*

The first part shows that the method of taking the smallest sphere intersecting $[\neg\phi]$, expounded by (Def $\ddot{-}$ from $\mathcal{S}$) is an accurate rendering of a severe withdrawal function. The second part states that the method for constructing systems of spheres via (Def $\mathcal{S}$ from $\ddot{-}$) or, equivalently (Def $\mathcal{S}$ from $\dot{-}$) by Lemma 11, does give a system of spheres for which the smallest sphere intersecting $[\neg\phi]$ corresponds to $K\ddot{-}\phi$.

This concludes our direct treatment of severe withdrawal in terms of systems of spheres. We shall return to systems of spheres in a slightly different context later.

10. Retrieving Epistemic Entrenchment Relations from Rational Withdrawals

While systems of spheres encode an ordering on worlds (consistent and complete sets of sentences), epistemic entrenchment orders sentences. In this section we concentrate on the methods used to generate an epistemic entrenchment (relative to K) from a given AGM contraction or severe withdrawal function (over K).

The fundamental idea of how to retrieve an entrenchment relation from belief change behaviour is this. A sentence ϕ is epistemically less entrenched in a belief state K than a sentence ψ if and only if an agent in belief state K who is forced to give up either ϕ or ψ will give up ϕ and hold on to ψ. This idea can be set in motion when we realise that to give up either ϕ or ψ can very well be rephrased as the task of giving up $\phi \wedge \psi$. So let a contraction function $\dot{-}$ (of any kind) be given.

(Def $<$ from $\dot{-}$) $\quad \phi < \psi$ iff $\psi \in K \dot{-} (\phi \wedge \psi)$ and $\phi \notin K \dot{-} (\phi \wedge \psi)$

The second clause is necessary since the agent may just refuse to withdraw $\phi \wedge \psi$. Rott [11, 30, 31] argues that it is indeed best to work with strict relations $<$ of epistemic entrenchment provided one is interested in having the flexibility to sensibly weaken the postulates involved (in particular, to drop the requirement that everything is comparable in terms of entrenchment) and in finding one-to-one correspondences between postulates for entrenchment and postulates relating to contraction behaviour or to rational choices. We shall not, however, pursue this project further here but keep to the original, more simple, if less flexible, account of Gärdenfors and Makinson. Thus we shall work with non-strict relations $\leq$ which may be thought of as the converse complements of the above-mentioned strict relations and we use the postulate ($\ddot{-}$4) to restrict refusal of contraction to logically true sentences. The following is the original definition of Gärdenfors and Makinson [9, p. 89].

(Def $\leq$ from $\dot{-}$) $\quad \phi \leq \psi$ iff $\phi \notin K \dot{-} (\phi \wedge \psi)$ or $\vdash \phi \wedge \psi$

As with systems of spheres, if $\ddot{-}$ is a severe withdrawal function, then the process of retrieving entrenchments from contractions can be simplified considerably.

(Def $\leq$ from $\ddot{-}$) $\quad \phi \leq \psi$ iff $\phi \notin K \ddot{-} \psi$ or $\vdash \psi$

This essentially means that the condition (Def $\ddot{-}$ from $\leq$) (see section 4) can be used in both directions. Except for some limiting cases, ψ is in $K \ddot{-} \phi$ if and only if $\phi < \psi$. This greatly simplifies the transition between severe withdrawal functions and their associated epistemic entrenchment relations.

We now show that the two conditions above are equivalent as far as severe withdrawals are concerned. Again, as in the case for systems of spheres, this is due to the properties induced by the postulates for severe withdrawal.

LEMMA 16. *If $\ddot{-}$ is a severe withdrawal function, then the two conditions* (Def $\leq$ from $\dot{-}$) *and* (Def $\leq$ from $\ddot{-}$) *are equivalent.*

In order to prove these and subsequent results, we recall the definition of epistemic entrenchment as introduced by Gärdenfors and Makinson [8, 9].

DEFINITION 5. Let $\leq$ be an ordering of the sentences of $\mathcal{L}$. We call $\leq$ a relation of *epistemic entrenchment* with respect to some belief set K, if it satisfies the following conditions:

(E1) If $\phi \leq \psi$ and $\psi \leq \chi$ then $\phi \leq \chi$ (Transitivity)

(E2) If $\phi \vdash \psi$ then $\phi \leq \psi$ (Dominance)

(E3) $\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$ (Conjunctiveness)

(E4) If $K \neq \mathcal{L}$ then: $\phi \leq \psi$ for every $\psi \in \mathcal{L}$ iff $\phi \notin K$ (Minimality)

(E5) If $\psi \leq \phi$ for every $\psi \in \mathcal{L}$, then $\vdash \phi$ (Maximality)

It follows from (E1) – (E5) that an epistemic entrenchment is a total preorder over sentences in which tautologies are greatest while non-beliefs are smallest elements. While an entrenchment ordering is an ordering of beliefs in K, systems of spheres can be seen as ordering worlds outside $[K]$. We shall return to the relationship between entrenchment and systems of spheres in a subsequent section.

The final result in this section lends further weight to our claim that the pairing of AGM contractions and severe withdrawals that we suggested in Section 7 is the right one. The result shows that any two functions related by the appropriate definitions generate identical relations of epistemic entrenchment.

OBSERVATION 17. *Let $\dot{-}$ and $\ddot{-}$ be corresponding AGM contraction and severe withdrawal functions either via (Def $\ddot{-}$ from $\dot{-}$) or via (Def $\dot{-}$ from $\ddot{-}$). Then $\dot{-}$ and $\ddot{-}$ lead to identical entrenchment relations, via (Def $\leq$ from $\dot{-}$) and (Def $\leq$ from $\ddot{-}$).*

11. Representation Theorems for Entrenchment-Based Withdrawals

In this section we turn to more technical results concerning the notion of epistemic entrenchment. In essence, we would like to formally show the appropriateness of (Def $\ddot{-}$ from $\leq$) and (Def $\leq$ from $\ddot{-}$) introduced in Sections 4 and 10 just as we were able to do for analogous definitions in terms of systems of spheres in Section 9. The following representation theorem is due to Gärdenfors and Makinson.[25]

OBSERVATION 18. [9, Theorems 4 and 5] *(i) If $\leq$ satisfies (E1) – (E5), then the function $\dot{-}$ obtained from $\leq$ by (Def $\dot{-}$ from $\leq$) is an AGM contraction function, that is, it satisfies ($\dot{-}$1) – ($\dot{-}$8).*
(ii) If $\dot{-}$ is an AGM contraction function, then $\dot{-}$ can be represented as an entrenchment-based contraction where the relation $\leq$ on which $\dot{-}$ is based is obtained by (Def $\leq$ from $\dot{-}$) *and $\leq$ satisfies (E1) – (E5).*

The first part states that the method of retaining ψ in contracting ϕ when $\phi \vee \psi$ is strictly more entrenched than ϕ gives an AGM entrenchment relation. The second part shows that the appropriate entrenchment relation can be obtained from an AGM contraction function using the recipe given by (Def $\leq$ from $\dot{-}$).

We can now formulate an entirely parallel representation theorem for severe withdrawals. This result is the epistemic entrenchment analogue of Observation 15 for systems of spheres.

OBSERVATION 19. *(i) If $\leq$ satisfies (E1) – (E5), then the function $\ddot{-}$ obtained from $\leq$ by* (Def $\ddot{-}$ from $\leq$) *is a severe withdrawal function.*
(ii) If $\ddot{-}$ is a severe withdrawal function, then $\ddot{-}$ can be represented as an entrenchment-based withdrawal where the relation $\leq$ on which $\ddot{-}$ is based is obtained by (Def $\leq$ from $\dot{-}$) *(or equivalently, by* (Def $\leq$ from $\ddot{-}$)*), and $\leq$ satisfies (E1) – (E5).*

The first part shows that the technique of retaining ψ whenever it is strictly more entrenched than ϕ, i.e., the technique expounded in (Def $\ddot{-}$ from $\leq$), gives a severe withdrawal function. The second part states that the method for constructing entrenchment relations via (Def $\leq$ from $\dot{-}$), or equivalently via (Def $\leq$ from $\ddot{-}$), gives an entrenchment relation for which the set of beliefs more entrenched than ϕ is $K \ddot{-} \phi$.

This result shows that we can use the same sort of entrenchment relation as Gärdenfors and Makinson but we apply it in a different manner which favours the Principles of Preference and Indifference over the Principle of Minimal Change — thereby violating Recovery. We can retain the same definition (Def $\leq$ from $\dot{-}$) as in the Gärdenfors-Makinson framework to reconstruct the underlying entrenchment relation from some observed severe withdrawal behaviour; in our framework, however, the definition can be simplified to (Def $\leq$ from $\ddot{-}$). Like Gärdenfors and Makinson for the case of AGM contractions, we obtain a perfect match between severe withdrawal functions and entrenchment relations.

12. Relating Spheres and Entrenchments

Up till now we have been studying AGM contractions and severe withdrawals from the point of view of both sphere semantics and entrenchment semantics; two important constructive modellings in the setting of AGM-style belief change. We have found that there is a far-reaching parallel between these two kinds of semantics or constructions for belief change functions. Now we want to give an explanation of that parallel in terms of a direct bridge between systems of spheres and entrenchment relations, bypassing any particular type of belief change function.

We begin by considering how to retrieve systems of spheres from epistemic entrenchment relations. Given some entrenchment relation $\leq$, we construct the corresponding system of spheres $\mathcal{S}(\leq)$ as follows (that is, we accumulate all such S_ϕ's and $\mathcal{M}_{\mathcal{L}}$ as in Section 8):

(Def $\mathcal{S}$ from $\leq$) $\quad S_\phi = [\{\psi : \phi < \psi\}]$

The set of sentences $\{\psi : \phi < \psi\}$ on the right-hand-side of (Def $\mathcal{S}$ from $\leq$) is a *cut* in the sense of [29, p. 159]. First we have to check whether we actually obtain a system of spheres from this construction.

LEMMA 20. *For any entrenchment relation $\leq$ with respect to K, the system of spheres $\mathcal{S}(\leq)$ satisfies conditions ($\mathcal{S}$1) – ($\mathcal{S}$4) with respect to $[K]$.*

Next we show that the system of spheres obtained from an entrenchment relation in this way is equivalent with the latter in the sense that it leads to the same AGM contraction and the same severe withdrawal function. More precisely, we show that the AGM contraction function (respectively, severe withdrawal function) obtained from a system of spheres $\mathcal{S}(\leq)$ derived from an entrenchment relation $\leq$ is the same as the AGM contraction function (respectively, severe withdrawal function) obtained directly from the entrenchment relation $\leq$.

OBSERVATION 21. *For any entrenchment relation $\leq$, the AGM contractions and the severe withdrawals generated from $\leq$ and $\mathcal{S}(\leq)$ are identical, i.e., $\mathcal{C}(\mathcal{S}(\leq)) = \mathcal{C}(\leq)$ and $\mathcal{W}(\mathcal{S}(\leq)) = \mathcal{W}(\leq)$.*

Here, $\mathcal{C}(\leq)$ refers to the AGM contraction function obtained from the entrenchment relation $\leq$ by means of (Def $\dot{-}$ from $\leq$). Similarly, $\mathcal{C}(\mathcal{S}(\leq))$ is the AGM contraction function obtained from the system of spheres $\mathcal{S}(\leq)$ via (Def $\dot{-}$ from $\mathcal{S}$). Again, $\mathcal{W}(\leq)$ and $\mathcal{W}(\mathcal{S}(\leq))$ refer to the severe withdrawal function obtained by the relevant definitions in Sections 3 and 4.

Let us now turn our attention to the reverse problem of obtaining an epistemic entrenchment relation from a system of spheres. Given some system of spheres $\mathcal{S}$, we construct the corresponding entrenchment relation $\leq = \mathcal{E}(\mathcal{S})$ as follows.

(Def $\leq$ from $\mathcal{S}$) $\quad \phi \leq \psi \quad$ iff $\quad$ for all $S \in \mathcal{S}$ if $S \subseteq [\phi]$ then $S \subseteq [\psi]$

We check whether we actually obtain an entrenchment relation from this construction.

LEMMA 22. *For any system of spheres $\mathcal{S}$ with respect to $[K]$, the entrenchment relation $\mathcal{E}(\mathcal{S})$ satisfies conditions (E1) – (E5) with respect to K.*

Given the nestedness ($\mathcal{S}1$) and the limit assumption ($\mathcal{S}4$) for systems of spheres, this condition reduces to the following.[26]

(Def$'$ $\leq$ from $\mathcal{S}$) $\quad \phi \leq \psi$ iff $c_{\mathcal{S}}(\neg\psi) \not\subseteq [\phi]$

We first show that this definition fits together with the one for the converse direction introduced above. The notation employed in the following observation should be self-explanatory by now.

OBSERVATION 23. *Let $\leq$ be an entrenchment relation and $\mathcal{S}$ a system of spheres. Then*
(i) $\mathcal{E}(\mathcal{S}(\leq)) = \leq$.
(ii) $\mathcal{S}(\mathcal{E}(\mathcal{S}))$ is the topological closure of the trimming of $\mathcal{S}$, i.e., $(\mathcal{S}_t)_{cl}$.

The first part of this result exposes a strong connection between epistemic entrenchment and systems of spheres. The second result, while not quite as strong, shows that applying (Def$'$ $\leq$ from $\mathcal{S}$) followed by (Def $\mathcal{S}$ from $\leq$) leads to an equivalent (although not necessarily identical — see Lemma 12) system of spheres. Together they indicate an isomorphism between epistemic entrenchment and a particular subclass (those trimmed and topologically closed) of systems of spheres.

We finally show the sphere analogue of Observation 21. That is, that the AGM contraction function (respectively, severe withdrawal function) obtained from an entrenchment relation $\mathcal{E}(\mathcal{S})$ derived from a system of spheres $\mathcal{S}$ is the same as that obtained directly from $\mathcal{S}$ itself.

OBSERVATION 24. *For any system of spheres $\mathcal{S}$, the AGM contractions and the severe withdrawals generated from $\mathcal{S}$ and $\mathcal{E}(\mathcal{S})$ are identical, i.e., $\mathcal{C}(\mathcal{E}(\mathcal{S})) = \mathcal{C}(\mathcal{S})$ and $\mathcal{W}(\mathcal{E}(\mathcal{S})) = \mathcal{W}(\mathcal{S})$.*

Taken together, these results demonstrate the appropriateness of the definitions introduced in this section.

13. Discussion

Levi [18] advocates a construction for belief removal based on *saturatable sets* rather than AGM's maximal consistent subsets of K not implying ϕ (denoted $K \perp \phi$). He notes that all elements K' of $K \perp \phi$ have the property that $Cn(K' \cup \{\neg\phi\})$ is a consistent complete theory (i.e., obey the *maxichoice* property). Yet, there are subsets of K not in $K \perp \phi$ also possessing this property. These sets Levi refers to as *saturatable sets* (the collection of which we denote $K \perp\!\!\!\perp \phi$ here). More precisely, $K' \in K \perp\!\!\!\perp \phi$ if and only if (*i*) $K' \subseteq K$, (*ii*) $\phi \notin K'$, and (*iii*) $Cn(K' \cup \{\neg\phi\})$ is a consistent complete theory. Hansson and Olsson [14] place this work in context with the AGM showing that a selection function applied to the set of saturatable sets generates a withdrawal function that satisfies postulates ($\dot{-}$1) – ($\dot{-}$4), ($\dot{-}$6) and Failure. In other words, this construction can be seen as capturing that of a withdrawal function satisfying the Failure property. They extend this work by showing that a selection function defined via a real-valued measure (satisfying a weak monotonicity condition) gives a construction satisfying the supplementary postulates ($\dot{-}7$) and ($\dot{-}8$). However, they do not supply a "completeness" result for this extended set of postulates. In light of the work presented here, severe withdrawal is a further restricted construction that can be given a complete characterisation. That is, a severe withdrawal represents an axiomatisable subclass of those belief removal operations characterised by Hansson and Olsson's real-valued measure selection function construction.

Let us, however, return to Levi's arguments on this subject. Levi uses the term *contraction* to denote any function removing, say, ϕ from K. In Makinson's [23] terminology which as adopted in our Definition 1 such functions are termed *withdrawals*; *contraction* being reserved for those withdrawals satisfying the additional Recovery postulate ($\dot{-}5$) and characterisable via meets of maximal non-implying subsets. The class of withdrawals can be obtained by taking meets of saturatable contractions removing ϕ but not meets of maximal subsets not implying ϕ. For this reason Levi maintains that one should consider meets of saturatable contractions rather than merely meets of maximal non-implying subsets. While AGM begin with the concept of a maximal non-implying subset as a way of achieving a minimal (in the sense of set inclusion) change in removing ϕ from K before settling on (partial) meets of such sets, Levi begins at the "other end." He embraces saturatable sets since meets of these will capture all withdrawal behaviour. Of course, admitting saturatable sets, and meets of them, violates Recovery (see Figures 4 and 5 in Appendix B) — a postulate Levi is strongly opposed to.

Now Levi's major concern in contraction follows the broad aims of the Principle of Minimal Change and, more specifically, the Principle of Informational Economy; that is, to minimise the loss of informational value. As such, it is important to specify how informational value is measured. Levi considers three different measures at various stages during the development of his ideas. Initially he considered *undamped (or probability-based) informational value* [18, p. 127] where the loss of informational value of the meet of a set of saturatable contractions is calculated using the sum of the losses of informational value of the minimal (in the sense of set inclusion) members of this set. He rejected this proposal immediately as it leads to a saturated contraction in every case and therefore satisfies the maxichoice property which both Levi and AGM agree is unreasonable. In its place he advocated *damped informational value (version 1)* [18, 20] in which intersections (meets) of saturatable contractions incur a loss of informational value equal to the largest loss incurred by a member of the set. However, Levi [20, p. 32] cites an example where he claims that a class of version 1 contractions which satisfy Recovery are counterintuitive. Moreover, Levi claims lack of uniformity in that damped informational value (version 1) equals undamped informational value in some cases but the two diverge in others. As a result, this was superseded by *damped informational value (version 2)* [19] where loss of informational value is minimised by taking the meet of those maximal subsets of K not implying ϕ with minimal undamped informational value and other saturatable contractions with (undamped) informational value no greater than this. In these latter two methods Levi adopts a Rule for Ties where "when two or more options tie for optimality one should adopt the intersection of all of them" [20, p. 27] with the proviso that such a "tie breaking" mechanism be adopted only when the resultant option is optimal. This last class of contraction functions are referred to as *mild contractions* by Levi [20]. It turns out that, when placed in a common setting, mild contractions coincide with severe withdrawals. Interestingly enough it has turned out, by our observations surrounding (Def$'$ $\ddot{-}$ from $\dot{-}$), that Levi could have captured "mild contractions" by considering meets of maximal non-implying subsets — although, to contract K by ϕ you would need to consider meets of certain maximal subsets of K not implying $\phi \wedge \psi$ for all $\psi \in \mathcal{L}$.

Levi [20] criticises our choice of terminology because it is based on a measure of loss in terms of subset inclusion (which we do not deny) and he maintains that informational value should not be measured in these terms; loss of damped informational value of type 2 is minimised and thus the contraction (or withdrawal) is mild. We wish to emphasise, however, that while our terminology is influenced by the fact that severe withdrawals tend to remove more beliefs than other revision equivalent proposals (see Observation 7), our arguments in favour of severe withdrawal in this paper are not motivated by this factor at all (nor, of course, by Informational Economy) but, rather, by the concerns of principled belief removal behaviour and, most of all, respecting of Indifference and Preference.

In an elegant paper, Kaluzhny and Lehmann [17] give a characterisation of nonmonotonic inference operations *Inf* for which $\mathit{Inf}(\Delta)$ can be represented as the set of all monotonic consequences together with some set $\mathit{Ass}(\Delta)$ of assumptions that are "compatible" with Δ: $\mathit{Inf}(\Delta) = \mathit{Cn}(\Delta \cup \mathit{Ass}(\Delta))$.[27] Their intuitive idea is that the assumption operator $\mathit{Ass}(\Delta)$ is *antitonic* in the sense that for $\Delta \subseteq \Gamma$ we get $\mathit{Ass}(\Gamma) \subseteq \mathit{Ass}(\Delta)$. The more premises, the less assumptions are compatible with them.

Given the well-known connections between nonmonotonic reasoning and belief revision (see for instance [10, 11]), it is easy to recognise that for finite $\Delta \subseteq \mathcal{L}$, Kaluzhny and Lehmann's assumption set $\mathit{Ass}(\Delta)$ corresponds to our severe withdrawal $K \ddot{-} (\neg \bigwedge \Delta)$, with $K = \mathit{Inf}(\emptyset)$ left implicit. Their condition of antitony is the analogue of our condition ($\ddot{-}$7a). The constructions for $\mathit{Ass}(\Delta)$ they use in their Theorems 2.1 and 2.2, viz. $\mathit{Ass}(\Delta) = \bigcap \{\mathit{Inf}(\Gamma) : \Gamma \subseteq \Delta\}$ and $\mathit{Ass}(\Delta) = \bigcap \{\mathit{Inf}(\Gamma) : \Gamma \subseteq \mathit{Cn}(\Delta)\}$ respectively, are reminiscent of our definition (Def$'$ $\ddot{-}$ from $\dot{-}$). But there are also important differences. They work on the level of postulates only without considering explicit constructions of nonmonotonic inference operations or the general principles that might motivate them. They work in contexts that do not validate the rule of Rational Monotony which corresponds to the belief revision postulate ($\dot{-}$8) alias $(\dot{-}8)$. And, perhaps most importantly, nonmonotonic inference relations correspond to *revisions* rather than removals of beliefs. Due to the revision equivalence of belief contractions and withdrawals, then, the distinction we are most interested in vanishes. To put it differently, Kaluzhny and Lehmann do not present a study of their *Ass* operation in its own right.

14. Conclusions

The AGM account of belief change is guided by principles of rationality. However, contrary to the popular perception given by the literature, the Principle of Informational Economy cannot be given unrestrained prominence over other rationality principles. It must be seen as only one of a number of factors to be taken into consideration when deciding which beliefs to discard. In fact, it works in combination with principles such as those of Indifference and Preference in this regard. Once this is accepted, it can be seen that AGM are in fact applying the latter principles only in so far as the $\neg\phi$-worlds are concerned and disregarding the ϕ-worlds. This position seems difficult to motivate and support. As a result, we propose a new form of belief removal operation, severe withdrawal, which applies these principles uniformly over all possible worlds. The contentious postulate of recovery is not satisfied by severe withdrawal.

In the present work we have attempted a comprehensive treatment of an alternative to AGM contraction which takes the Principles of Indifference and Preference into account; we call this severe withdrawal. We showed how these principles point toward a different way of using two important AGM constructions: systems of spheres and epistemic entrenchment. In these constructions the objects to which the principles are applied are, in the first case, worlds (or models) and, in the second, sentences of the object language. Both methods lead to simple mechanisms for constructing removals of belief.

Interestingly enough, if one prefers to focus on belief revision rather then belief removal, then the effects, via the Levi identity, are unnoticeable. That is, in any revision equivalent class of withdrawal functions there will be exactly one AGM contraction function [23] and one severe withdrawal function. Severe withdrawal functions can be seen as setting a lower bound on interesting withdrawal behaviour within each of these revision equivalent classes. We furnished a way of moving backwards and forwards between the corresponding AGM contraction function and severe withdrawal function in a given class. We also supplied methods for obtaining the desired severe withdrawal behaviour from the constructive modellings of systems of spheres and epistemic entrenchment relations. Furthermore, mechanisms for going back the other way — extracting the relevant underlying structure (total pre-order on worlds or one on sentences) — were given. It is interesting to note in regard to this latter point that the definitions for AGM contraction can be used for severe withdrawal to achieve the same effect but, in general, may be simplified. Finally methods were given for mapping directly between systems of spheres and epistemic entrenchment relations that lead to the same AGM contraction or severe withdrawal function.

One last, and important, moral can be drawn from this exposition with regard to the constructive modellings. Clearly the underlying structure (a systems of spheres or an epistemic entrenchment relation) is important in achieving belief removal (or belief change in general for that matter). However, the way we use this structure is also very crucial. Starting with a fixed structure, different principles give rise to different behaviour. More importantly, this behaviour, through the principles that bring it about, can be motivated by rational means. Here, the Principles of Indifference and Preference — arguably rational integrity constraints — lead to severe withdrawal.

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Appendix

A. Proofs

We reproduce the following properties of $th : 2^{\mathcal{M}_{\mathcal{L}}} \to \mathcal{K}$, listed by Grove [12], for reference. They will be useful for some of the proofs that follow.

LEMMA 0. *Properties of* th [12].

(i) $th([K]) = K$ *for all belief sets (i.e., theories)* K *if the underlying logic is compact*

(ii) $th(X) \neq K_{\perp}$ *if and only if* X *is nonempty*

(iii) For any sentence $\phi \in \mathcal{L}$ and $X \subseteq \mathcal{M}_{\mathcal{L}}$, $th(X \cap [\phi]) = Cn\,(th(X) \cup \{\phi\})$

(iv) For $X,\ X' \subseteq \mathcal{M}_{\mathcal{L}}$, if $X \subseteq X'$, then $th(X') \subseteq th(X)$

(v) For $K,\ K' \in \mathcal{K}$, if $K \subseteq K'$, then $[K'] \subseteq [K]$

LEMMA 1. *Let the basic postulates ($\ddot{-}$1) – ($\ddot{-}$4) and ($\ddot{-}$6) be given. Then*
(i) ($\ddot{-}$7a) and ($\ddot{-}$8) taken together are equivalent with ($\ddot{-}$9);
(ii) ($\ddot{-}$7a) and ($\ddot{-}$8) taken together imply ($\ddot{-}$10).

Proof. Assume that the basic postulates ($\ddot{-}$1) – ($\ddot{-}$4) and ($\ddot{-}$6) are satisfied.

(i) ($\ddot{-}$9) implies ($\ddot{-}$7a): Let $\nvdash \phi$. Then by ($\ddot{-}$4), $\phi \notin K \ddot{-} \phi$, so by ($\ddot{-}$1), $\phi \wedge \psi \notin K \ddot{-} \phi$. Hence, by ($\ddot{-}$9), $K \ddot{-} \phi \subseteq K \ddot{-} (\phi \wedge \psi)$.

($\ddot{-}$9) implies ($\ddot{-}$8): This is immediate on substituting $\phi \wedge \psi$ for ψ.

($\ddot{-}$7a) and ($\ddot{-}$8) imply ($\ddot{-}$9): Let $\phi \notin K \ddot{-} \psi$. With the help of ($\dot{-}$8c), $\phi \notin K \ddot{-} (\phi \wedge \psi)$. Hence, by ($\ddot{-}$8), $K \ddot{-} (\phi \wedge \psi) \subseteq K \ddot{-} \phi$. But by ($\ddot{-}$7a), $K \ddot{-} \psi \subseteq K \ddot{-} (\phi \wedge \psi)$ whenever $\nvdash \psi$ and hence $K \ddot{-} \psi \subseteq K \ddot{-} \phi$ as desired. If $\vdash \phi$, then $K \ddot{-} \psi = K$ by ($\ddot{-}$2) and ($\ddot{-}$3). Now $K \subseteq K \ddot{-} \phi$ by ($\ddot{-}$2) and therefore $K \ddot{-} \psi \subseteq K \ddot{-} \phi$ trivially.

(ii) Let $\nvdash \phi$ and $\phi \in K \ddot{-} \psi$. If $\vdash \psi$, then $K \subseteq K \ddot{-} \psi$ by ($\ddot{-}$3), so $K \ddot{-} \phi \subseteq K \ddot{-} \psi$ follows from ($\ddot{-}$2). So let ψ be such that $\nvdash \psi$. From $\phi \in K \ddot{-} \psi$ and ($\ddot{-}$7a), we conclude that $\phi \in K \ddot{-} (\phi \wedge \psi)$. Since $\nvdash \phi \wedge \psi$, ($\ddot{-}$1) and ($\ddot{-}$4) give us $\psi \notin K \ddot{-} (\phi \wedge \psi)$. Hence, by ($\ddot{-}$8), $K \ddot{-} (\phi \wedge \psi) \subseteq K \ddot{-} \psi$. On the other hand, by ($\ddot{-}$7a), $K \ddot{-} \phi \subseteq K \ddot{-} (\phi \wedge \psi)$. Hence $K \ddot{-} \phi \subseteq K \ddot{-} \psi$, as desired. □

LEMMA 2. *Let $\ddot{-}$ be a severe withdrawal function over K. Then*
(i) Either $K \ddot{-} \phi \subseteq K \ddot{-} \psi$ or $K \ddot{-} \psi \subseteq K \ddot{-} \phi$.
(ii) Either $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \phi$ or $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \psi$.
(iii) If $K \ddot{-} \phi \wedge \psi \subseteq K \ddot{-} \psi$, then $\psi \notin K \ddot{-} \phi$ or $\vdash \phi$ or $\vdash \psi$.
(iv) If $\nvdash \phi$ and $\nvdash \psi$, then either $\phi \notin K \ddot{-} \psi$ or $\psi \notin K \ddot{-} \phi$.

Proof. (i) Consider two cases: (a) $\phi \notin K \ddot{-} \psi$ and (b) $\phi \in K \ddot{-} \psi$ In the former case ($\ddot{-}$9) gives $K \ddot{-} \psi \subseteq K \ddot{-} \phi$. In the latter case, if $\nvdash \phi$, then ($\ddot{-}$10) gives $K \ddot{-} \phi \subseteq K \ddot{-} \psi$. Otherwise, $\vdash \phi$ and by ($\ddot{-}$3) $K \subseteq K \ddot{-} \phi$, and by ($\ddot{-}$2) $K \ddot{-} \psi \subseteq K$ so $K \ddot{-} \psi \subseteq K \ddot{-} \phi$.

(ii) Using ($\dot{-}$7c) and ($\dot{-}$8c) it is easily seen that if $\psi \in K \ddot{-} (\phi \wedge \psi)$, then $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \phi$, and if $\phi \in K \ddot{-} (\phi \wedge \psi)$, then $K \ddot{-} (\phi \wedge \psi) = K \ddot{-} \psi$. A similar situation holds for $\phi \in K \ddot{-} (\phi \wedge \psi)$. Consider then, the case where $\phi,\ \psi \notin K \ddot{-} (\phi \wedge \psi)$. By two applications of ($\ddot{-}$8), $K \ddot{-} (\phi \wedge \psi) \subseteq K \ddot{-} \phi$ and $K \ddot{-} (\phi \wedge \psi) \subseteq K \ddot{-} \psi$. Now consider two further subcases: (a) at least one of $\nvdash \phi$ or $\nvdash \psi$ holds, and (b) both $\vdash \phi$ and $\vdash \psi$ hold. In the former case, either $K \ddot{-} \phi \subseteq K \ddot{-} (\phi \wedge \psi)$ or $K \ddot{-} \psi \subseteq K \ddot{-} (\phi \wedge \psi)$ holds by ($\ddot{-}$7a). It follows

that either $K\ddot{-}\phi = K\ddot{-}(\phi \wedge \psi)$ or $K\ddot{-}\psi = K\ddot{-}(\phi \wedge \psi)$. In the latter case, $K\ddot{-}\phi = K\ddot{-}\psi = K\ddot{-}(\phi \wedge \psi)$, by ($\ddot{-}$2) and ($\ddot{-}$3).

(iii) Let $K\ddot{-}\phi \wedge \psi \subseteq K\ddot{-}\psi$ and $\nvdash \phi$ and $\nvdash \psi$. Suppose for reductio that $\psi \in K\ddot{-}\phi$. Then by ($\ddot{-}$7a) $\psi \in K\ddot{-}(\phi \wedge \psi)$. So since $K\ddot{-}\phi \wedge \psi \subseteq K\ddot{-}\psi$, we also get $\psi \in K\ddot{-}\psi$, contradicting ($\ddot{-}$4).

(iv) Let $\nvdash \phi$ and $\nvdash \psi$. Suppose for contradiction that both $\phi \in K\ddot{-}\psi$ and $\psi \in K\ddot{-}\phi$. Then by ($\ddot{-}$10), $K\ddot{-}\phi = K\ddot{-}\psi$, so $\phi \in K\ddot{-}\phi$ and $\psi \in K\ddot{-}\psi$, contradicting ($\ddot{-}$4). □

LEMMA 3. *There is no contraction function over a non-trivial belief set K that satisfies postulates ($\dot{-}$1) – ($\dot{-}$8) and ($\ddot{-}$9).*

Proof. Suppose there is a contraction function $\dot{-}$ over K that satisfies all of ($\dot{-}$1) – ($\dot{-}$8) and ($\ddot{-}$9). Suppose further that K is non-trivial, i.e., that there is a ϕ such that $\phi \in K \setminus Cn(\emptyset)$ and $K \not\subseteq Cn(\phi)$. We first show that $K\dot{-}\phi = Cn(\emptyset)$. Suppose, for *reductio ad absurdum*, that there is a $\psi \notin Cn(\emptyset)$ such that $\psi \in K\dot{-}\phi$. Now $\psi \in K$ by ($\dot{-}$2). It follows by ($\ddot{-}$10), which we showed to follow from ($\dot{-}$1) – ($\dot{-}$8) and ($\ddot{-}$9) (in Lemma 1*(i)* and *(ii)*), that $K\dot{-}\psi \subseteq K\dot{-}\phi$. By ($\dot{-}$5), ($\dot{-}$1) and the Deduction Theorem $\psi \rightarrow \phi \in K\dot{-}\psi \subseteq K\dot{-}\phi$. However, by ($\dot{-}$4), $\phi \notin K\dot{-}\phi$ so by ($\dot{-}$1), $\psi \notin K\dot{-}\phi$ contradicting our initial supposition. Therefore $K\dot{-}\phi = Cn(\emptyset)$. Consequently, $K \not\subseteq Cn((K\dot{-}\phi) \cup \{\phi\}) = Cn(\phi)$ violating recovery ($\dot{-}$5). □

LEMMA 4. *Let $\dot{-}$ and $\dot{-}'$ be two withdrawal functions that are revision equivalent. Then $\dot{-}$ and $\dot{-}'$ are identical whenever either of the following two clauses holds:*

(i) $\dot{-}$ and $\dot{-}'$ satisfy ($\dot{-}$1), ($\dot{-}$2) and Recovery ($\dot{-}$5);

(ii) $\dot{-}$ and $\dot{-}'$ are severe withdrawal functions.

Proof. Let $\dot{-}$ and $\dot{-}'$ be revision equivalent withdrawal functions.

(i) Let $\dot{-}$ and $\dot{-}'$ satisfy ($\dot{-}$1), ($\dot{-}$2) and Recovery ($\dot{-}$5). We need to show that $\dot{-} = \dot{-}'$. Left to right inclusion. Suppose $\psi \in K\dot{-}\phi$. We need to show that $\psi \in K\dot{-}'\phi$ as well. First we show that $\neg\phi \rightarrow \psi \in K\dot{-}'\phi$. From $\psi \in K\dot{-}\phi$, we conclude using monotonicity of *Cn* and the Levi identity that $\psi \in Cn((K\dot{-}\phi) \cup \{\neg\phi\}) = K * \neg\phi$. By revision equivalence, we get $\psi \in K *' (\neg\phi) = Cn((K\dot{-}'\phi) \cup \{\neg\phi\})$, so by the Deduction Theorem for *Cn* and ($\dot{-}$1), $\neg\phi \rightarrow \psi \in K\dot{-}'\phi$. On the other hand, we know from ($\dot{-}$2) that $\psi \in K\dot{-}\phi \subseteq K$. So by ($\dot{-}$5) and ($\dot{-}$1), $\phi \rightarrow \psi$ is in $K\dot{-}'\phi$. From this and the previously established fact that $\neg\phi \rightarrow \psi$ is in $K\dot{-}'\phi$, we conclude with ($\dot{-}$1) that ψ is in fact in $K\dot{-}'\phi$, as desired.

The right to left inclusion can be proved in the same fashion with $\dot{-}$ and $\dot{-}'$ exchanged.

(ii) Let $\ddot{-}$ and $\ddot{-}'$ be severe withdrawal functions. We need to show that $\ddot{-} = \ddot{-}'$. Left to right inclusion. Suppose $\psi \in K \ddot{-} \phi$. We need to show that $\psi \in K \ddot{-}' \phi$ as well. If $\phi \in Cn(\emptyset)$, then $K \ddot{-} \phi = K = K \ddot{-}' \phi$, by ($\ddot{-}$2) and ($\ddot{-}$3). So let $\phi \notin Cn(\emptyset)$. Then it follows from $\psi \in K \ddot{-} \phi$ that $\psi \in K \ddot{-} (\phi \wedge \psi)$, by ($\ddot{-}$7a). Thus also $\psi \in Cn(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\}) = K * \neg(\phi \wedge \psi)$, using the monotonicity of *Cn* and the Levi identity. By revision equivalence, then $\psi \in K *' \neg(\phi \wedge \psi) = Cn(K \ddot{-}' (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$. By the Deduction Theorem for *Cn* and ($\ddot{-}$1), we get that $\neg(\phi \wedge \psi) \rightarrow \psi \in K \ddot{-}' (\phi \wedge \psi)$, which means, by ($\ddot{-}$1) again, that $\psi \in K \ddot{-}' (\phi \wedge \psi)$. Using ($\dot{-}$8c) (or alternatively, ($\ddot{-}$4) and ($\ddot{-}$8)), we get that $K \ddot{-}' (\phi \wedge \psi) \subseteq K \ddot{-}' \phi$ and thus $\psi \in K \ddot{-}' \phi$, as desired.

The right to left inclusion can be proved in the same fashion with $\ddot{-}$ and $\ddot{-}'$ exchanged. $\square$

LEMMA 5. (Def $\ddot{-}$ from $\dot{-}$) *and* (Def$'$ $\dot{-}$ from $\ddot{-}$) *are equivalent.*

Proof. It is sufficient to show, for every $\phi \in \mathcal{L}$ such that $\nvdash \phi$, that $\chi \in K \dot{-} (\phi \wedge \chi)$ iff $\chi \in \bigcap\{K \dot{-} (\phi \wedge \psi) : \psi \in \mathcal{L}\}$.

Right to left is trivial. Let $\chi \in \bigcap\{K \dot{-} (\phi \wedge \psi) : \psi \in \mathcal{L}\}$. Consequently, $\chi \in K \dot{-} (\phi \wedge \psi)$ for all $\psi \in \mathcal{L}$. Choosing $\psi \equiv \chi$ we get $\chi \in K \dot{-} (\phi \wedge \chi)$ as desired.

From left to right, let $\chi \in K \dot{-} (\phi \wedge \chi)$. We need to show $\chi \in \bigcap\{K \dot{-} (\phi \wedge \psi) : \psi \in \mathcal{L}\}$. We can do so by showing that $\chi \in K \dot{-} (\phi \wedge \psi)$ for arbitrary $\psi \in \mathcal{L}$. Now $\chi \vee \neg\phi \vee \neg\psi \in K \dot{-} (\phi \wedge \chi) = K \dot{-} ((\chi \vee \neg\phi \vee \neg\psi) \wedge ((\phi \wedge \chi) \vee (\phi \wedge \psi)))$ using ($\dot{-}$1) for the former part and ($\dot{-}$6) for the latter. It follows by ($\dot{-}$8c) and ($\dot{-}$6) that $K \dot{-} (\phi \wedge \chi) \subseteq K((\phi \wedge \chi) \vee (\phi \wedge \psi))$. Therefore, $\chi \in K \dot{-} ((\phi \wedge \chi) \vee (\phi \wedge \psi))$. From our initial supposition and ($\dot{-}$2), $\chi \in K$ giving by ($\dot{-}$5) and ($\dot{-}$1) that $(\psi \vee \neg\chi) \rightarrow \chi \in K \dot{-} (\psi \vee \neg\chi)$. Consequently $\chi \in K \dot{-} (\psi \vee \neg\chi)$ by ($\dot{-}$1). It therefore follows by ($\dot{-}$7) and our previous reasoning that $\chi \in K \dot{-} (((\phi \wedge \chi) \vee (\phi \wedge \psi)) \wedge (\psi \vee \neg\chi))$. Hence, by ($\dot{-}$6) $\chi \in K \dot{-} (\phi \wedge \psi)$ as desired. $\square$

OBSERVATION 6. *If $\dot{-}$ is an AGM contraction function, then $\ddot{-}$ as obtained by* (Def $\ddot{-}$ from $\dot{-}$) *is a severe withdrawal function revision equivalent to $\dot{-}$, and $K \ddot{-} \phi \subseteq K \dot{-} \phi$ for all $\phi \in \mathcal{L}$.*

Proof. Let $\dot{-}$ be an AGM contraction function and $\ddot{-}$ be obtained from $\dot{-}$ via (Def $\ddot{-}$ from $\dot{-}$). We first show that $\ddot{-}$ is a severe withdrawal function. (By Lemma 5, (Def $\ddot{-}$ from $\dot{-}$) and (Def$'$ $\ddot{-}$ from $\dot{-}$) are equivalent so we can make use of both definitions to simplify the proof.)

($\ddot{-}$1) If $\nvdash \phi$, then $K\ddot{-}\phi = \bigcap\{K\dot{-}(\phi \wedge \psi) : \psi \in \mathcal{L}\}$ by (Def$'$ $\ddot{-}$ from $\dot{-}$). Since $K\dot{-}(\phi \wedge \psi)$ is a theory for every $\psi \in \mathcal{L}$ by ($\dot{-}$1), then clearly $K\ddot{-}\phi$ is too. Otherwise, $\vdash \phi$ in which case $K\ddot{-}\phi = K$ and again $K\ddot{-}\phi$ is a theory.

($\ddot{-}$2) If $\nvdash \phi$, then $K\ddot{-}\phi = \bigcap\{K\dot{-}(\phi \wedge \psi) : \psi \in \mathcal{L}\}$ by (Def$'$ $\ddot{-}$ from $\dot{-}$). Since $K\dot{-}(\phi\wedge\psi) \subseteq K$ for all $\psi \in \mathcal{L}$ by ($\dot{-}$2) clearly $K\ddot{-}\phi \subseteq K$. Otherwise, $\vdash \phi$ and by (Def$'$ $\ddot{-}$ from $\dot{-}$) $K\ddot{-}\phi = K$ therefore $K\ddot{-}\phi \subseteq K$ trivially.

($\ddot{-}$3) If $\vdash \phi$, $K\ddot{-}\phi = K$ by (Def $\ddot{-}$ from $\dot{-}$) and the desired result follows trivially. Otherwise, $\nvdash \phi$ and $\phi \notin K$. Then $K\ddot{-}\phi = \bigcap\{K\dot{-}(\phi \wedge \psi) : \psi \in \mathcal{L}\}$ by (Def $\ddot{-}$ from $\dot{-}$). Since $\phi \notin K$, then $\phi \wedge \psi \notin K$ for all $\psi \in \mathcal{L}$. Therefore $K \subseteq K\dot{-}(\phi \wedge \psi)$ for all $\psi \in \mathcal{L}$ by ($\dot{-}$3). Hence $K \subseteq K\ddot{-}\phi$ as desired.

($\ddot{-}$4) Let $\nvdash \phi$. Now $\phi \notin K\dot{-}\phi$ by ($\dot{-}$4). It follows by ($\dot{-}$6) that $\phi \notin K\dot{-}(\phi \wedge \phi)$. Therefore, $\phi \notin K\ddot{-}\phi$ by (Def $\ddot{-}$ from $\dot{-}$).

($\ddot{-}$6) Follows trivially using ($\dot{-}$6).

($\ddot{-}$7a) Let $\nvdash \phi$. Suppose $\chi \in K\ddot{-}\phi$. Then via (Def $\ddot{-}$ from $\dot{-}$) $\chi \in K\dot{-}(\phi\wedge\chi)$. It follows by ($\dot{-}$7) that $\chi \in K\dot{-}((\phi\wedge\chi)\wedge\psi)$. (Actually this last part follows more directly from condition ($\dot{-}$P) $K\dot{-}\phi \cap \mathit{Cn}\,(\phi) \subseteq K\dot{-}(\phi\wedge\psi)$ — with $\phi \equiv \phi \wedge \psi$ and $\psi \equiv \psi$ — which is equivalent to ($\dot{-}$7) [1, Observation 3.3 p. 516]). Hence $\chi \in K\dot{-}\phi \wedge \psi$ by ($\dot{-}$6) and (Def $\ddot{-}$ from $\dot{-}$).

($\ddot{-}$8) Let $\phi \notin K\ddot{-}(\phi \wedge \psi)$. We need to show that $K\ddot{-}(\phi \wedge \psi) \subseteq K\ddot{-}\phi$. If $\vdash \phi$, then $K\ddot{-}\phi = K$ by (Def $\ddot{-}$ from $\dot{-}$) and $K\ddot{-}(\phi\wedge\psi) \subseteq K$ by ($\ddot{-}$2) which was shown above to hold. Therefore, it follows directly that $K\ddot{-}(\phi \wedge \psi) \subseteq K\ddot{-}\phi$. Otherwise $\nvdash \phi$. Therefore, by (Def $\ddot{-}$ from $\dot{-}$) $K\ddot{-}(\phi \wedge \psi) = \{\chi : \chi \in K\dot{-}((\phi \wedge \psi) \wedge \chi)\}$ and $K\ddot{-}\phi = \{\chi : \chi \in K\dot{-}(\phi \wedge \chi)\}$. Suppose $\chi \in K\ddot{-}(\phi \wedge \psi)$. We need to show that $\chi \in K\ddot{-}\phi$ and can do so by showing that $\chi \in K\dot{-}(\phi\wedge\chi)$. Since $\chi \in K\ddot{-}(\phi\wedge\psi)$ we have $\chi \in K\dot{-}((\phi\wedge\psi)\wedge\chi)$ $(*)$ by (Def $\ddot{-}$ from $\dot{-}$). It follows that $\phi\wedge\psi \notin K\dot{-}((\phi\wedge\psi)\wedge\chi)$ by ($\dot{-}$4) and ($\dot{-}$8) subsequently gives $K\dot{-}((\phi\wedge\psi)\wedge\chi) \subseteq K\dot{-}(\phi\wedge\psi)$ (#). Our initial assumption that $\phi \notin K\ddot{-}(\phi \wedge \psi)$ and (Def $\ddot{-}$ from $\dot{-}$) give $\phi \notin K\dot{-}((\phi \wedge \psi) \wedge \phi)$ which by ($\dot{-}$6) means $\phi \notin K\dot{-}(\phi \wedge \psi)$. Using the contrapositive of (#) we get that $\phi \notin K\dot{-}((\phi \wedge \psi) \wedge \chi)$ and ($\dot{-}$1) then gives $\phi \wedge \chi \notin K\dot{-}((\phi \wedge \psi) \wedge \chi)$. Applying ($\dot{-}$8) again (and an application of ($\dot{-}$6) to the left-hand-side) we see that $K\dot{-}((\phi \wedge \psi) \wedge \chi) \subseteq K\dot{-}(\phi \wedge \chi)$. It therefore follows from $(*)$ that $\chi \in K\dot{-}(\phi \wedge \chi)$ as required.

We now show that $\dot{-}$ and $\ddot{-}$ are revision equivalent. That is, we show that $K*\phi = K\ddot{*}\phi$. Now $K*\phi = \mathit{Cn}\,(K\dot{-}\neg\phi \cap \{\phi\})$ and $K\ddot{*}\phi = \mathit{Cn}\,(K\ddot{-}\neg\phi \cap \{\phi\})$ by the Levi identity. We first prove left to right holds. Suppose $\chi \in K*\phi$. Then $\phi \to \chi \in K\dot{-}\neg\phi$ by the Levi identity, Deduction Theorem and ($\dot{-}$1). Now $\vdash [\neg\phi \wedge (\phi \to \chi)] \leftrightarrow \neg\phi$ so by ($\dot{-}$6), $\phi \to \chi \in K\dot{-}(\neg\phi \wedge (\phi \to \chi))$ $(*)$. We consider two cases : (a) $\nvdash \neg\phi$; and, (b) $\vdash \neg\phi$. In the former case, it follows by (Def $\ddot{-}$ from $\dot{-}$) and $(*)$ that $\phi \to \chi \in K\ddot{-}\neg\phi$. Using the Deduction Theorem, $\chi \in \mathit{Cn}\,(K\ddot{-}\neg\phi \cup \{\phi\})$. Hence $\chi \in K\ddot{*}\phi$ as required. In the latter case $K\dot{-}\neg\phi = K = K\ddot{-}\neg\phi$ by ($\dot{-}$1) and ($\dot{-}$5) and (Def $\ddot{-}$from $\dot{-}$) respectively. It follows that $\phi \to \chi \in K\ddot{-}\neg\phi$ and consequently $\chi \in K\ddot{*}\phi$

via applications of the Deduction Theorem and the Levi identity as required. Right to left is similar.

It remains to show that $K \ddot{-} \phi \subseteq K \dot{-} \phi$. This follows straightforwardly from the result by Makinson [23, Observation p. 389] however we include a proof in terms of our own definitions. Suppose $\chi \in K \ddot{-} \phi$. If $\vdash \phi$, then $K \ddot{-} \phi = K = K \dot{-} \phi$ by (Def $\ddot{-}$ from $\dot{-}$) for the former part and ($\dot{-}$1) and ($\dot{-}$5) for the latter and the result follows trivially. Otherwise $\nvdash \phi$. Now $\chi \in K \dot{-} (\phi \wedge \chi)$ by (Def $\ddot{-}$ from $\dot{-}$). By ($\dot{-}$8c) $K \dot{-} (\phi \wedge \chi) \subseteq K \dot{-} \phi$. Therefore $\chi \in K \dot{-} \phi$ as desired. □

OBSERVATION 7. *Let $\dot{-}$ be an AGM contraction function. Then the severe withdrawal function $\ddot{-}$ defined from $\dot{-}$ by definition* (Def $\ddot{-}$ from $\dot{-}$) *is the smallest withdrawal function in terms of set-theoretic inclusion satisfying postulate ($\dot{-}$8c) which is revision equivalent to $\dot{-}$.*

Proof. Let $\dot{-}$ be an AGM contraction function and $\ddot{-}$ defined from $\dot{-}$ via (Def $\ddot{-}$ from $\dot{-}$). Let $-$ be any withdrawal function satisfying ($\dot{-}$8c) which is revision equivalent to $\dot{-}$ (and therefore $\ddot{-}$ also by Observation 6). We need to show that $K \ddot{-} \phi \subseteq K - \phi$.

Suppose $\psi \in K \ddot{-} \phi$. If $\vdash \phi$, then $K \ddot{-} \phi = K \dot{-} \phi = K$ (the former by ($\ddot{-}$3) which is satisfied by Observation 6 and the latter by ($\dot{-}$1) and ($\dot{-}$5)). Since $-$ satisfies Failure $K - \phi = K$ and $\psi \in K - \phi$ as desired. Otherwise, $\nvdash \phi$. By (Def $\ddot{-}$ from $\dot{-}$), $\psi \in K \dot{-} (\phi \wedge \psi)$ so by the Levi identity $\neg\phi \wedge \psi \in K \dot{*} \neg(\phi \wedge \psi) = K \ddot{*} \neg(\phi \wedge \psi)$. By the revision equivalence of $\dot{-}$ and $-$ (and $\ddot{-}$), $\neg\phi \wedge \psi \in K * \neg(\phi \wedge \psi)$ where $*$ is defined from $-$ via the Levi identity. Using the Levi identity again, $\neg(\phi \wedge \psi) \rightarrow (\neg\phi \wedge \psi) \in K - (\phi \wedge \psi)$. That is, by ($\dot{-}$1), $\psi \in K - (\phi \wedge \psi)$. But then by ($\dot{-}$8c), $K - (\phi \wedge \psi) \subseteq K - \phi$. Hence $\psi \in K - \phi$ as required. □

OBSERVATION 8. *Let $\dot{-}$ be an AGM contraction function. Then the withdrawal function $\dddot{-}$ defined from $\dot{-}$ by definition* (Def $\dddot{-}$ from $\dot{-}$) *is the smallest withdrawal function which is revision equivalent to $\dot{-}$.*

Proof. Let $\dot{-}$ be an AGM contraction function and $\dddot{-}$ defined from $\dot{-}$ via (Def $\dddot{-}$ from $\dot{-}$). We first verify that $\dddot{-}$ is a withdrawal function (i.e., satisfies ($\dot{-}$1) – ($\dot{-}$4) and ($\dot{-}$5)).

($\dot{-}$1) If $\phi \in K$ and $\nvdash \phi$, we have $K \dddot{-} \phi = Cn\,(\phi) \cap K \dot{-} \phi$ by (Def $\dddot{-}$ from $\dot{-}$) which is obviously closed by ($\dot{-}$1) and the properties of Cn. Otherwise, $\phi \notin K$ or $\vdash \phi$ by (Def $\dddot{-}$ from $\dot{-}$). Again $K \dddot{-} \phi$ is closed.

($\dot{-}$2) If $\phi \notin K$ or $\vdash \phi$ (Def $\dddot{-}$ from $\dot{-}$) gives $K \dddot{-} \phi = K$ in which case the desired result follows trivially. Otherwise, $\phi \in K$ and $\nvdash \phi$ so $K \dddot{-} \phi =$

$Cn\,(\phi) \cap K \dot{-} \phi$ by (Def $\dddot{-}$ from $\dot{-}$). Now by ($\dot{-}$2) $K \dot{-} \phi \subseteq K$ so clearly $Cn\,(\phi) \cap K \dot{-} \phi \subseteq K$ and therefore $K \dddot{-} \phi \subseteq K$.

($\dot{-}$3) Let $\phi \notin K$. By (Def $\dddot{-}$ from $\dot{-}$) $K \dddot{-} \phi = K$ and $K \subseteq K \dddot{-} \phi$ follows trivially.

($\dot{-}$4) Let $\nvdash \phi$. If $\phi \notin K$, then by (Def $\dddot{-}$ from $\dot{-}$) $K \dddot{-} \phi = K$ so $\phi \notin K \dddot{-} \phi$. Otherwise, $\phi \in K$ and $K \dddot{-} \phi = Cn\,(\phi) \cap K \dot{-} \phi$ by (Def $\dddot{-}$ from $\dot{-}$). However, $\phi \notin K \dot{-} \phi$ by ($\dot{-}$2) and therefore $\phi \notin Cn\,(\phi) \cap K \dot{-} \phi = K \dddot{-} \phi$.

($\dot{-}$6) Let $\vdash \phi \leftrightarrow \psi$. If $\phi \notin K$ and $\vdash \phi$ then clearly $\psi \notin K$ and $\vdash \psi$. By (Def $\dddot{-}$ from $\dot{-}$) we have $K \dddot{-} \phi = K = K \dddot{-} \psi$ as desired. Otherwise $\phi \in K$ and $\nvdash \phi$. Clearly then $\psi \in K$ and $\nvdash \psi$. Now $K \dddot{-} \phi = Cn\,(\phi) \cap K \dot{-} \phi$ and $K \dddot{-} \psi = Cn\,(\psi) \cap K \dot{-} \psi$. Moreover, $Cn\,(\phi) = Cn\,(\psi)$ by our supposition at the outset and $K \dot{-} \phi = K \dot{-} \psi$ by ($\dot{-}$6). Hence $K \dddot{-} \phi = Cn\,(\phi) \cap K \dot{-} \phi = Cn\,(\psi) \cap K \dot{-} \psi = K \dddot{-} \psi$ as desired.

(Note: it is easily shown that $\dddot{-}$ satisfies Failure also.)

Next we show that $\dddot{-}$ and $\dot{-}$ are revision equivalent. Left to right. Suppose $\chi \in K \dddot{*} \phi$. Then $\phi \to \chi \in K \dddot{-} \neg\phi$ by the Levi identity, Deduction Theorem and ($\dot{-}$1) (which has been shown above to hold). If $\neg\phi \notin K$ or $\vdash \neg\phi$, then $K \dddot{-} \neg\phi = K$ by (Def $\dddot{-}$ from $\dot{-}$) and $K \dot{-} \neg\phi = K$ by ($\dot{-}$2) and ($\dot{-}$3)/($\dot{-}$1) and ($\dot{-}$5). Therefore, $\phi \to \chi \in K \dot{-} \neg\phi$ and $\phi \in K * \phi$ by the Deduction Theorem and the Levi identity. Otherwise, $\neg\phi \in K$ and $\vdash \neg\phi$. By (Def $\dddot{-}$ from $\dot{-}$), $K \dddot{-} \neg\phi = Cn\,(\neg\phi) \cap K \dot{-} \neg\phi$ and again it follows that $\phi \to \chi \in K \dot{-} \neg\phi$ whereby we proceed as above.

Right to left. Suppose $\chi \in K * \neg\phi$. Then $\phi \to \chi \in K \dot{-} \neg\phi$ by the Levi identity, Deduction Theorem and ($\dot{-}$1). If $\neg\phi \notin K$ or $\vdash \neg\phi$, then $K \dddot{-} \neg\phi = K = K \dot{-} \neg\phi$ as above and therefore $\phi \to \chi \in K \dddot{-} \neg\phi$ whereby the Deduction Theorem and the Levi identity give $\phi \in K \dddot{*} \phi$. Otherwise, $\neg\phi \in K$ and $\vdash \neg\phi$. Now clearly $\phi \to \chi \in Cn\,(\neg\phi)$. So $\phi \to \chi \in Cn\,(\neg\phi) \cap K \dot{-} \neg\phi$ and by (Def $\dddot{-}$ from $\dot{-}$) $\phi \to \chi \in K \dddot{-} \neg\phi$. Hence by the Deduction Theorem and the Levi identity $\chi \in K \dddot{-} \phi$ as desired.

Finally, we show that $\dddot{-}$ is the smallest withdrawal function revision equivalent to $\dot{-}$. Suppose $\psi \in K \dddot{-} \phi$. We need to show that $\psi \in K - \phi$ for any withdrawal function $-$ revision equivalent to $\dot{-}$. Now $\psi \in K$ by ($\dot{-}$2) which $\dddot{-}$ was shown to satisfy above. If $\vdash \phi$ or $\phi \notin K$, then $K \dot{-} \phi = K$ by ($\dot{-}$1) and ($\dot{-}$5)/($\dot{-}$2) and ($\dot{-}$3). Since $-$ satisfies Failure and ($\dot{-}$2) and ($\dot{-}$3) (since it is a withdrawal function) $K - \phi = K$ and $\psi \in K - \phi$ as desired. Otherwise, $\nvdash \phi$ or $\phi \notin K$. By (Def $\dddot{-}$ from $\dot{-}$), $\psi \in Cn\,(\phi) \cap K \dot{-} \phi$. Therefore, $\psi \in Cn\,(\phi)$ (i.e., $\phi \vdash \psi$) and $\psi \in K \dot{-} \phi$. It follows that $\neg\phi \wedge \psi \in K * \neg\phi = Cn\,(K \dot{-} \phi \cup \{\neg\phi\})$. The revision equivalence of of $\dot{-}$ and $-$ (and $\dddot{-}$) gives $\neg\phi \wedge \psi \in K * \neg\phi$. That is, $\neg\phi \wedge \psi \in Cn\,(K - \phi \cup \{\neg\phi\})$ and the Deduction Theorem and ($\dot{-}$1) give $\neg\phi \to (\neg\phi \wedge \psi) \in K - \phi$. By ($\dot{-}$1) again we obtain $\phi \vee \psi \in K - \phi$ but since $\phi \vdash \psi$ we have $\psi \in K - \phi$ as required. $\square$

OBSERVATION 9. *If* $\dddot{-}$ *is a severe withdrawal function, then* $\dot{-}$ *as obtained by* (Def $\dot{-}$ from $\dddot{-}$) *is an AGM contraction function, and* $K \dddot{-} \phi \subseteq K \dot{-} \phi$ *for* *all* $\phi \in \mathcal{L}$.

Proof. Let $\ddot{-}$ be a severe withdrawal function and $\dot{-}$ be obtained from $\ddot{-}$ via (Def $\dot{-}$ from $\ddot{-}$).

We first show that $\dot{-}$ is an AGM contraction function.

($\dot{-}1$) In the case that $\nvdash \phi$, we have $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ by (Def $\dot{-}$ from $\ddot{-}$) which is obviously closed by the properties of Cn. Otherwise, $\vdash \phi$ and $K \dot{-} \phi = K$ by (Def $\dot{-}$ from $\ddot{-}$). Again, $K \dot{-} \phi$ is closed.

($\dot{-}2$) If $\vdash \phi$, then $K \dot{-} \phi = K$ by (Def $\dot{-}$ from $\ddot{-}$) and so obviously $K \dot{-} \phi \subseteq K$. Otherwise, $\nvdash \phi$ and (Def $\dot{-}$ from $\ddot{-}$) gives $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. Clearly $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\}) \subseteq K$ so $K \dot{-} \phi \subseteq K$.

($\dot{-}3$) Let $\phi \in K$. If $\vdash \phi$, then $K \dot{-} \phi = K$ by (Def $\dot{-}$ from $\ddot{-}$) and $K \subseteq K \dot{-} \phi$. Otherwise, $\vdash \phi$ and $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. Now by ($\ddot{-}3$) we have $K \subseteq K \ddot{-} \phi$ and therefore, by monotonicity of Cn, $K \subseteq Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. Hence $K \subseteq K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ and consequently $K \subseteq K \dot{-} \phi$ as desired.

($\dot{-}4$) Let $\nvdash \phi$. Then $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ by (Def $\dot{-}$ from $\ddot{-}$). Suppose $\phi \in K \dot{-} \phi$. Then $\phi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ by the monotonicity of Cn. By the Deduction Theorem, and ($\ddot{-}1$) $\neg\phi \to \phi \in K \ddot{-} \phi$ or, again by ($\ddot{-}1$), $\phi \in K \ddot{-} \phi$ contradicting ($\ddot{-}4$). It follows that $\phi \notin K \dot{-} \phi$.

($\dot{-}5$) If $\nvdash \phi$, then $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. By (Def $\dot{-}$ from $\ddot{-}$). Suppose for reductio ad absurdum that there is a $\psi \in K$ such that $\psi \notin Cn\,(K \dot{-} \phi \cup \{\phi\})$. That is, $\psi \notin Cn\,((K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})) \cup \{\phi\})$. By the Deduction Theorem $\phi \to \psi \notin Cn\,(K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\}))$. Now either $\phi \to \psi \notin K$ or $\phi \to \psi \notin Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. In the former case we have an immediate contradiction since $\psi \in K$ implies $\phi \to \psi \in K$. In the latter case we have $\neg\phi \to (\phi \to \psi) \notin K \ddot{-} \phi$ by the Deduction Theorem and ($\ddot{-}1$). Equivalently $\top \notin K \ddot{-} \phi$ contradicting ($\ddot{-}1$). Otherwise $\vdash \phi$ in which case $K \dot{-} \phi = K$ and $K \subseteq Cn\,(K \cup \{\phi\}) = Cn\,(K \dot{-} \phi \cup \{\phi\})$ by monotonicity.

($\dot{-}6$) Let $\vdash \phi \leftrightarrow \psi$. Suppose $\nvdash \phi$, then $\nvdash \psi$. We have by (Def $\dot{-}$ from $\ddot{-}$) that $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\}) = K \cap Cn\,(K \ddot{-} \psi \cup \{\neg\psi\}) = K \dot{-} \psi$. Otherwise, $\nvdash \phi$ implying $\nvdash \psi$ and $K \dot{-} \phi = K = K \dot{-} \psi$ by (Def $\dot{-}$ from $\ddot{-}$).

($\dot{-}7$) Suppose that $\nvdash \phi$ and $\nvdash \psi$. We need to show that $K \dot{-} \phi \cap K \dot{-} \psi \subseteq K \dot{-} (\phi \wedge \psi)$. By (Def $\dot{-}$ from $\ddot{-}$) we have the following: $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \cup \{\neg\phi\})$, $K \dot{-} \psi = K \cap Cn\,(K \ddot{-} \cup \{\neg\psi\})$ and $K \dot{-} (\phi \wedge \psi) = K \cap Cn\,(K \ddot{-} \cup \{\neg(\phi \wedge \psi)\})$. Suppose $\chi \in K \dot{-} \phi \cap K \dot{-} \psi$. Then $\chi \in K \dot{-} \phi$ and $\chi \in K \dot{-} \psi$. So $\chi \in K$ by ($\ddot{-}2$) and $\chi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ and $\chi \in Cn\,(K \ddot{-} \psi \cup \{\neg\psi\})$ by the monotonicity of Cn. By the Deduction Theorem and ($\ddot{-}1$) $\neg\phi \to \chi \in K \ddot{-} \phi$ and $\neg\psi \to \chi \in K \ddot{-} \psi$. We need to show that $\chi \in K \dot{-} (\phi \wedge \psi)$. We can do so by showing that $\chi \in K$ and $\chi \in Cn\,(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$ (i.e., by the Deduction Theorem, $\neg(\phi \wedge \psi) \to \chi \in K \ddot{-} (\phi \wedge \psi)$ or, equivalently, $(\neg\phi \to \chi) \wedge (\neg\psi \to \chi) \in K \ddot{-} (\phi \wedge \psi)$). By ($\ddot{-}7$a) and our reasoning above we have that $\neg\phi \to \chi \in K \ddot{-} (\phi \wedge \psi)$

and $\neg\psi \rightarrow \chi \in K \ddot{-} (\phi \wedge \psi)$. Therefore, by ($\ddot{-}$1), $(\neg\phi \rightarrow \chi) \wedge (\neg\psi \rightarrow \chi) \in K \ddot{-} (\phi \wedge \psi)$ as desired.

Otherwise, at least one of $\vdash \phi$, or $\vdash \psi$ holds. If both $\vdash \phi$ and $\vdash \psi$ hold and therefore $\vdash \phi \wedge \psi$, (Def $\dot{-}$ from $\ddot{-}$) gives $K \dot{-} \phi \cap K \dot{-} \psi = K \cap K = K = K \dot{-} (\phi \wedge \psi)$ with the result holding trivially. So assume only one of $\vdash \phi, \vdash \psi$ holds. Without loss of generality, suppose $\vdash \phi$ and $\not\vdash \psi$. Now $K \dot{-} \phi = K$ by (Def $\dot{-}$ from $\ddot{-}$). It also follows by ($\dot{-}$2), which we have shown above to hold, that $K \dot{-} \psi \subseteq K$. Therefore, $K \dot{-} \phi \cap K \dot{-} \psi = K \cap K \dot{-} \psi = K \dot{-} \psi \subseteq (\phi \wedge \psi)$ (the last of these by ($\dot{-}$7a)).

($\dot{-}$8) Let $\phi \notin K \dot{-} (\phi \wedge \psi)$. If $\vdash \phi \wedge \psi$, then (Def $\dot{-}$ from $\ddot{-}$) gives $K \dot{-} (\phi \wedge \psi) = K$. It follows by our initial assumption that $\phi \notin K$. But this contradicts the fact that $\vdash \phi \wedge \psi$ so this case is not possible. Otherwise $\not\vdash \phi \wedge \psi$. Moreover, we can assume that $\not\vdash \phi$ for, otherwise, $K \dot{-} \phi = K$ by (Def $\dot{-}$ from $\ddot{-}$) and the result follows directly via ($\dot{-}$2) which was shown above to hold. Suppose now that $\chi \in K \dot{-} (\phi \wedge \psi)$. By (Def $\dot{-}$ from $\ddot{-}$) we have $K \dot{-} (\phi \wedge \psi) = K \cap Cn\,(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$. This latter fact, together with the Deduction Theorem and ($\ddot{-}$1) give $\neg(\phi \wedge \psi) \rightarrow \chi \in K \ddot{-} (\phi \wedge \psi)$ or, in other words (again appealing to ($\ddot{-}$1)) $(\neg\phi \rightarrow \chi) \wedge (\neg\psi \rightarrow \chi) \in K(\phi \wedge \psi)$(#). We also know that either $\phi \notin K$ or $\phi \notin Cn\,(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$ using the assumption at the outset of this proof. The former does not hold under our current assumptions so the latter must hold and, via the Deduction Theorem and ($\ddot{-}$1), we have $\neg(\phi \wedge \psi) \rightarrow \phi \notin K \ddot{-} (\phi \wedge \psi)$ or, in other words $\phi \notin K \ddot{-} (\phi \wedge \psi)$. Now ($\ddot{-}$8) and (#) give $(\neg\phi \rightarrow \chi) \wedge (\neg\psi \rightarrow \chi) \in K \ddot{-} \phi$. So, in particular $\neg\phi \rightarrow \chi \in K \ddot{-} \phi$ by ($\ddot{-}$1) and the Deduction Theorem gives $\chi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. Since $\chi \in K$ we have via (Def $\dot{-}$ from $\ddot{-}$) that $\chi \in K \dot{-} \phi$ as required.

We now show that $\dot{-}$ is revision equivalent to $\ddot{-}$. That is, we show that $K * \phi = K \ddot{*} \phi$. Now $K * \phi = Cn\,(K \dot{-} \neg\phi \cup \{\phi\})$ and $K \ddot{*} \phi = Cn\,(K \ddot{-} \neg\phi \cup \{\phi\})$ Left to right. Suppose $\chi \in K * \phi = Cn\,(K \dot{-} \neg\phi \cup \{\phi\})$. Now $\phi \rightarrow \chi \in K \dot{-} \neg\phi$ by the Deduction Theorem and ($\dot{-}$1) which was shown above to hold. We consider cases (a) $\vdash \neg\phi$; and, (b) $\not\vdash \neg\phi$. In the former case, by (Def $\ddot{-}$ from $\dot{-}$) $K \dot{-} \neg\phi = K$ and by ($\dot{-}$1) and ($\dot{-}$5) $K \ddot{-} \neg\phi = K$. Clearly then $\phi \rightarrow \chi \in K \ddot{-} \neg\phi$ and by the Deduction Theorem $\chi \in Cn\,(K \ddot{-} \neg\phi \cup \{\phi\}) = K \ddot{*} \phi$. In the latter case, we have $\phi \rightarrow \chi \in K \dot{-} \neg\phi$ by the Levi identity and ($\dot{-}$1). Right to left. Now suppose $\chi \in K \ddot{*} \phi = Cn\,(K \ddot{-} \neg\phi \cup \{\phi\})$. By the Deduction Theorem and ($\ddot{-}$1) $\phi \rightarrow \chi \in K \ddot{-} \neg\phi$. We consider two cases (a) $\vdash \neg\phi$; and, (b) $\vdash \neg\phi$. In the former case, by (Def $\dot{-}$ from $\ddot{-}$) $K \dot{-} \neg\phi = K$ and by ($\ddot{-}$3) $K \ddot{-} \neg\phi = K$. Clearly $\phi \rightarrow \chi \in K \dot{-} \neg\phi$ and the Deduction Theorem gives $\chi \in Cn\,(K \dot{-} \phi \cup \{\phi\}) = K * \phi$.

Now, in the latter case, by (Def $\dot{-}$ from $\ddot{-}$) $K \dot{-} \neg\phi = K \cap Cn\,(K \ddot{-} \neg\phi \cup \{\phi\})$. By the monotonicity of Cn, $\phi \rightarrow \chi \in Cn\,(K \ddot{-} \neg\phi \cup \{\phi\})$ and $\phi \rightarrow \chi \in K$. Therefore, $\phi \rightarrow \chi \in K \dot{-} \neg\phi$ by ($\dot{-}$2). Hence $\chi \in Cn\,(K \dot{-} \neg\phi \cup \{\phi\}) = K * \phi$ by the Deduction Theorem and the Levi identity as desired.

It remains to show that $K \ddot{-} \phi \subseteq K \dot{-} \phi$ for all ϕ. This follows straightforwardly from Makinson's [23, Observation p. 389] result. However, we include a proof in terms of our definitions here. Consider two cases. (a) $\vdash \phi$; and (b) $\nvdash \phi$. In the former case, by (Def $\dot{-}$ from $\ddot{-}$) we have $K \dot{-} \phi = K$ and the result follows straightforwardly by ($\ddot{-}$2). In the latter case, by (Def $\dot{-}$ from $\ddot{-}$) we have $K \dot{-} \phi = K \cap (K \ddot{-} \phi \cup \{\neg\phi\})$. Now suppose $\chi \in K \ddot{-} \phi$. Clearly $\chi \in K$ by ($\ddot{-}$2) and $\chi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ by monotonicity of Cn. It therefore follows that $\chi \in K \dot{-} \phi$. $\square$

OBSERVATION 10. *(i) If we start with an AGM contraction function $\dot{-}$, turn it into a severe withdrawal function $\ddot{-}$ by* (Def $\ddot{-}$ from $\dot{-}$) *and turn the latter into an AGM contraction function $\dot{-}'$ by* (Def $\dot{-}$ from $\ddot{-}$)*, then we end up with $\dot{-}' = \dot{-}$.*

(ii) If we start with a severe withdrawal function $\ddot{-}$, turn it into an AGM contraction function $\dot{-}$ by (Def $\dot{-}$ from $\ddot{-}$) *and turn the latter into a severe withdrawal function $\ddot{-}'$ by* (Def $\ddot{-}$ from $\dot{-}$)*, then we end up with $\ddot{-}' = \ddot{-}$.*

Proof. Let $\dot{-}$, $\ddot{-}$ and $\dot{-}'$ be defined as in the statement above.

(i) Left to right. Suppose $\chi \in K \dot{-} \phi$. We need to show that $\chi \in K \dot{-}' \phi$. If $\vdash \phi$, then $K \dot{-} \phi = K \ddot{-} \phi = K$ and $K \dot{-}' \phi = K$ by ($\dot{-}$1) and ($\dot{-}$5), (Def $\ddot{-}$ from $\dot{-}$) and (Def $\dot{-}$ from $\ddot{-}$) respectively. Otherwise $\nvdash \phi$. By (Def $\ddot{-}$ from $\dot{-}$) $K \ddot{-} \phi = \{\psi : \psi \in K \dot{-} (\phi \wedge \psi)\}$ and by (Def $\dot{-}$ from $\ddot{-}$) $K \dot{-}' \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$. Since $\chi \in K \dot{-} \phi$ we have $\phi \vee \chi \in K \dot{-} \phi$ by ($\dot{-}$1). Using ($\dot{-}$6) we have $\phi \vee \chi \in K \dot{-} (\phi \vee (\phi \wedge \chi))$ and so $\phi \vee \chi \in K \ddot{-} \phi$ by (Def $\ddot{-}$ from $\dot{-}$). That is, by ($\ddot{-}$1) (which is satisfied by Observation 6), $\neg\phi \rightarrow \chi \in K \ddot{-} \phi$. The Deduction Theorem gives $\chi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ and by ($\ddot{-}$2) $\chi \in K$. Hence $\chi \in K \dot{-}' \phi$ by (Def $\dot{-}$from $\ddot{-}$) as desired.

Right to left. Suppose $\chi \in K \dot{-}' \phi$. If $\vdash \phi$ we can reason exactly as above. Otherwise $\nvdash \phi$. Now $\chi \in K$ and $\chi \in Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ by (Def $\dot{-}$ from $\ddot{-}$). As a result of applying the Deduction Theorem and ($\ddot{-}$1) we have $\neg\phi \rightarrow \chi \in K \ddot{-} \phi$. Therefore $\neg\phi \rightarrow \chi \in K \dot{-} (\phi \wedge (\neg\phi \rightarrow \psi)) = K \dot{-} \phi$ (the former part by (Def $\ddot{-}$ from $\dot{-}$) and the latter part by ($\dot{-}$6)). But ($\dot{-}$5) and ($\dot{-}$1) give $\phi \rightarrow \chi \in K \dot{-} \phi$. Putting these together we get by ($\dot{-}$1) that $\chi \in K \dot{-} \phi$ as required.

(ii) Let $\ddot{-}$, $\dot{-}$ and $\ddot{-}'$ be defined as in the statement above. If $\vdash \phi$ we reason along the lines of *(i)*. Otherwise, $\nvdash \phi$. Left to right. Suppose $\chi \in K \ddot{-} \phi$. By (Def $\dot{-}$ from $\ddot{-}$) $K \dot{-} \phi = K \cap Cn\,(K \ddot{-} \phi \cup \{\neg\phi\})$ and by (Def $\ddot{-}$ from $\dot{-}$) $K \ddot{-}' \phi = \{\chi : \chi \in K \dot{-} (\phi \wedge \psi)\}$. Now clearly $\chi \in K$ by ($\ddot{-}$2). So we know that $\phi \wedge \chi \in K$ and $\nvdash \phi \wedge \chi$. We need to show that $\chi \in K \ddot{-}' \phi$ which we can do, according to (Def $\ddot{-}$ from $\dot{-}$), by showing that $\chi \in K \dot{-} (\phi \wedge \chi)$. This we can do, according to (Def $\dot{-}$ from $\ddot{-}$) by showing that $\chi \in K \cap Cn\,(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$. We have already shown that $\chi \in K$ so it remains to show that $\chi \in Cn\,(K \ddot{-} (\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$ or equivalently, by the Deduction

Theorem and ($\ddot{-}$1), that $\neg(\phi \wedge \chi) \to \chi \in K \ddot{-} (\phi \wedge \psi)$. That is, by ($\ddot{-}$1) again, $\chi \in K \ddot{-} (\phi \wedge \psi)$. Since $\nvdash \phi$ and $\chi \in K \ddot{-} \phi$, this fact follows by ($\ddot{-}$7a).

Right to left. Suppose $\chi \in K \ddot{-}' \phi$. Now $\chi \in K \dot{-} (\phi \wedge \psi)$ by (Def $\ddot{-}$ from $\dot{-}$). By (Def $\dot{-}$ from $\ddot{-}$) $\chi \in K$ and $\chi \in Cn\,(K \ddot{-} (\phi \wedge \chi) \cup \{\neg(\phi \wedge \chi)\})$. The latter gives $\neg(\phi \wedge \chi) \to \chi \in K \ddot{-} (\phi \wedge \chi)$ by the Deduction Theorem and ($\ddot{-}$1). Therefore $\chi \in K \ddot{-} (\phi \wedge \chi)$. Hence by ($\ddot{-}$4) $\phi \notin K \ddot{-} (\phi \wedge \chi)$ and by ($\ddot{-}$8) $\chi \in K \ddot{-} \phi$ as required. □

LEMMA 11. *If* $\ddot{-}$ *is a severe withdrawal function, then the two conditions* (Def $\mathcal{S}$ from $\dot{-}$) *and* (Def $\mathcal{S}$ from $\ddot{-}$) *are equivalent.*

Proof. In the case where $\vdash \phi$ both constructions give the sphere $[K]$. (For the right-hand-side use ($\ddot{-}$2) and ($\ddot{-}$3) and Lemma 0*(v)*). Therefore, we have to show that

$$\bigcup\{[K \ddot{-} \psi] : [\psi] \subseteq [\phi]\} = [K \ddot{-} \phi]$$

whenever $\nvdash \phi$.

Left to right. Suppose $m \in \bigcup\{[K \ddot{-} \psi] : [\psi] \subseteq [\phi]\}$. Then $m \in [K \ddot{-} \psi]$ for some $[\psi] \subseteq [\phi]$. We need to show that $m \in [K \ddot{-} \phi]$. By ($\ddot{-}$4) $\phi \notin K \ddot{-} \phi$. Now since $[\psi] \subseteq [\phi]$ we have $\psi \vdash \phi$ so, by ($\ddot{-}$1), $\psi \notin K \ddot{-} \phi$. Using ($\ddot{-}$9) we have $K \ddot{-} \phi \subseteq K \ddot{-} \psi$. In other words, $[K \ddot{-} \psi] \subseteq [K \ddot{-} \phi]$ by Lemma 0*(v)* as desired.

Right to left. Suppose $m \in [K \ddot{-} \phi]$. We need to show $m \in [K \ddot{-} \psi]$ for some $[\psi] \subseteq [\phi]$. The result follows directly by choosing ψ to be ϕ. □

LEMMA 12. *$\mathcal{S}$, $\mathcal{S}_t$, $\mathcal{S}_u$ and $\mathcal{S}_{cl}$ are all equivalent.*

Proof. If $\vdash \phi$, then by (Def $c_{\mathcal{S}}$) we have $c_{\mathcal{S}}(\neg\phi) = c_{\mathcal{S}_t}(\neg\phi) = c_{\mathcal{S}_u}(\neg\phi) = c_{\mathcal{S}_{cl}}(\neg\phi) = [K]$ and by (Def $\ddot{-}$ from $\mathcal{S}$) $K \ddot{-}_{\mathcal{S}} \phi = K \ddot{-}_{\mathcal{S}_t} \phi = K \ddot{-}_{\mathcal{S}_u} \phi = K \ddot{-}_{\mathcal{S}_{cl}} \phi$. This fact can also be used to show $f_{\mathcal{S}}(\neg\phi) = f_{\mathcal{S}_t}(\neg\phi) = f_{\mathcal{S}_u}(\neg\phi) = f_{\mathcal{S}_{cl}}(\neg\phi)$ and so, by (Def $\dot{-}$ from $\mathcal{S}$), $K \dot{-}_{\mathcal{S}} \phi = K \dot{-}_{\mathcal{S}_t} \phi = K \dot{-}_{\mathcal{S}_u} \phi = K \dot{-}_{\mathcal{S}_{cl}} \phi$.

Therefore we consider the case where $\nvdash \phi$. Now by ($\mathcal{S}$4) (and (Def $c_{\mathcal{S}}$)) $c_{\mathcal{S}}(\neg\phi)$ exists. It follows directly by the definition of $\mathcal{S}_t$ that $c_{\mathcal{S}}(\neg\phi) = c_{\mathcal{S}_t}(\neg\phi)$.

We now want to show that $c_{\mathcal{S}}(\neg\phi) = c_{\mathcal{S}_u}(\neg\phi)$. Suppose to the contrary. Without loss of generality, since any sphere in $\mathcal{S}$ is also in $\mathcal{S}_u$ by definition, suppose that $c_{\mathcal{S}}(\neg\phi) \subset c_{\mathcal{S}_u}(\neg\phi)$. That is, there is some $S \in \mathcal{S}_u$ such that $S \cap [\neg\phi] \neq \emptyset$ and $S \subset c_{\mathcal{S}_u}$. By definition of $\mathcal{S}_u$ this means that there is some $S' \subseteq S$ and $S' \in \mathcal{S}$ and $S' \subset c_{\mathcal{S}}(\neg\phi)$. But this contradicts the definition of $c_{\mathcal{S}}(\neg\phi)$ via (Def $c_{\mathcal{S}}$). Hence $c_{\mathcal{S}}(\neg\phi) = c_{\mathcal{S}_u}(\neg\phi)$. By (Def $\ddot{-}$ from $\mathcal{S}$) and using Lemma 0*(iv)* we get $K \ddot{-}_{\mathcal{S}} \phi = K \ddot{-}_{\mathcal{S}_t} \phi = K \ddot{-}_{\mathcal{S}_u} \phi$. We can show $K \dot{-}_{\mathcal{S}} \phi = K \dot{-}_{\mathcal{S}_t} \phi = K \dot{-}_{\mathcal{S}_u} \phi$. in similar fashion.

It now remains to consider $K \dot{-}_{\mathcal{S}_{cl}} \phi$ and $K \ddot{-}_{\mathcal{S}_{cl}} \phi$. We begin by showing that $th([th(S)]) = th(S)$ for $S \subseteq \mathcal{M}_{\mathcal{L}}$ (*). Right to left. Suppose $\alpha \in th(S)$. Now $[th(S)] = \{m \in \mathcal{M}_{\mathcal{L}} : th(S) \subseteq m\}$. Since $\alpha \in the(S)$, then $\alpha \in m$ for all $m \in [th(S)]$. Therefore $\alpha \in \bigcap[th(S)]$ and hence $\alpha \in th([th(S)])$ as desired. Left to right. Suppose $\alpha \in th([th(S)])$. Further, suppose for reduction that $\alpha \notin th(S)$. Then there is some $m \in \mathcal{M}_{\mathcal{L}}$ such that $th(S) \subseteq m$ and $\neg\alpha \in m$ by Lindenbaum's lemma. It follows that $m \in [th(S)]$. Consequently, $\alpha \notin \bigcap[th(S)] = th([th(S)])$ contradicting our initial supposition. Hence $\alpha \in th(S)$ as desired.

Now $th(f_{\mathcal{S}_cl}(\neg\phi)) = th([(c_{\mathcal{S}}(\neg\phi))] \cap [\neg\phi])$ by the definition of $\mathcal{S}cl$ and the definition of $f_{\mathcal{S}}$. By Lemma 0*(iii)* $th([(c_{\mathcal{S}}(\neg\phi))] \cap [\neg\phi]) = Cn\,(th([th(c_{\mathcal{S}}(\neg\phi))]) \cup \{\neg\phi\})$. Using (*) we have that $th([th(c_{\mathcal{S}}(\neg\phi))]) = th(c_{\mathcal{S}}(\neg\phi))$. Therefore $Cn\,(th([th(c_{\mathcal{S}}(\neg\phi))]) \cup \{\neg\phi\}) = Cn\,(th(c_{\mathcal{S}}(\neg\phi)) \cup \{\neg\phi\})$ and by Lemma 0*(iii)* again $Cn\,(th(c_{\mathcal{S}}(\neg\phi)) \cup \{\neg\phi\}) = th(c_{\mathcal{S}}(\neg\phi) \cap [\neg\phi])$. But this latter part is just $th(f_{\mathcal{S}}(\neg\phi))$. Therefore $th(f_{\mathcal{S}_cl}(\neg\phi)) = th(f_{\mathcal{S}}(\neg\phi))$. We want to show $th([K] \cup f_{\mathcal{S}}(\neg\phi)) = th([K] \cup f_{\mathcal{S}_cl}(\neg\phi))$. Left to right. Now suppose $\psi \in th([K] \cup f_{\mathcal{S}}(\neg\phi))$. Then $\psi \in \bigcap([K] \cup f_{\mathcal{S}}(\neg\phi))$ by definition. That is, $\psi \in m$ for all m$\in [K] \cup f_{\mathcal{S}}(\neg\phi)$ so $\psi \in m$ for all $m \in [K]$ and $\psi \in m'$ for all $m' \in f_{\mathcal{S}}(\neg\phi)$. It follows that $\psi \in th(f_{\mathcal{S}}(\neg\phi))$. Consequently, by the above, $th(f_{\mathcal{S}_cl}(\neg\phi))$. Therefore $\psi \in m$ for all $m \in f_{\mathcal{S}_cl}(\neg\phi)$ and it follows that $\psi \in m$ for all $m \in [K] \cup f_{\mathcal{S}_cl}(\neg\phi)$. As a result $\psi \in th([K] \cup f_{\mathcal{S}_cl}(\neg\phi))$. Right to left is proved similarly. Hence by (Def $\dot{-}$ from $\mathcal{S}$) $K \dot{-}_{\mathcal{S}_cl} \phi = K \dot{-}_{\mathcal{S}} \phi$. Together with the results above we now have $K \dot{-}_{\mathcal{S}} \phi = K \dot{-}_{\mathcal{S}_t} \phi = K \dot{-}_{\mathcal{S}_u} \phi = K \dot{-}_{\mathcal{S}_cl} \phi$.

Now $th(c_{\mathcal{S}_cl}(\neg\phi)) = th([th(c_{\mathcal{S}}))])$ and using (*), as above, we have $th([th(c_{\mathcal{S}}))]) = th(c_{\mathcal{S}})$. Therefore $th(c_{\mathcal{S}_cl}(\neg\phi)) = th(c_{\mathcal{S}}(\neg\phi))$ and by (Def $\ddot{-}$ from $\mathcal{S}$) we get $K \ddot{-}_{\mathcal{S}_cl} \phi = K \ddot{-}_{\mathcal{S}} \phi$. Together with the results above we now have $K \ddot{-}_{\mathcal{S}} \phi = K \ddot{-}_{\mathcal{S}_t} \phi = K \ddot{-}_{\mathcal{S}_u} \phi = K \ddot{-}_{\mathcal{S}_cl} \phi$.

□

OBSERVATION 13. *Let $\dot{-}$ and $\ddot{-}$ be corresponding AGM contraction and severe withdrawal functions either via (Def $\ddot{-}$ from $\dot{-}$) or via (Def $\dot{-}$ from $\ddot{-}$). Then $\dot{-}$ and $\ddot{-}$ lead to equivalent systems of spheres, via (Def $\mathcal{S}$ from $\dot{-}$) and (Def $\mathcal{S}$ from $\ddot{-}$). More precisely, the system of spheres obtained from $\ddot{-}$ is the topological closure of that obtained from $\dot{-}$.*

Proof. Consider the second Lewis-Grovean construction of $\mathcal{S}(\dot{-})$, given by

$$\text{(Def}' \ \mathcal{S} \text{ from } \dot{-}) \qquad X_\phi = \begin{cases} \bigcup\{[K \dot{-} (\phi \wedge \psi)] : \psi \in \mathcal{L}\} & \text{whenever } \not\vdash \phi \\ [K] & \text{otherwise} \end{cases}$$

and compare it with the construction we get using (Def $\mathcal{S}$ from $\ddot{-}$) and (Def $\ddot{-}$ from $\dot{-}$):

$$\mathcal{S}(\mathcal{W}(\dot{-})): \quad X'_\phi = [K\ddot{-}\phi] = \begin{cases} [\bigcap\{K\dot{-}(\phi\wedge\psi) : \psi\in\mathcal{L}\}] & \text{whenever } \not\vdash \phi \\ [K] & \text{otherwise} \end{cases}$$

In order to show that $\mathcal{S}(\mathcal{W}(\dot{-}))$ is the topological closure of $\mathcal{S}(\dot{-})$, we prove for each ϕ that X'_ϕ is the topological closure of X_ϕ, i.e., $X'_\phi = [th(X_\phi)]$. Our claim is that

$$[\bigcap\{K\dot{-}(\phi\wedge\psi) : \psi\in\mathcal{L}\}] = [th(\bigcup\{[K\dot{-}(\phi\wedge\psi)] : \psi\in\mathcal{L}\})]$$

Now m is in the right-hand side iff m satisfies $\bigcap(\bigcup\{[K\dot{-}(\phi\wedge\psi)] : \psi\in\mathcal{L}\})$. This means that

(i) m satisfies all χ which are satisfied by all m' that are in $[K\dot{-}(\phi\wedge\psi)]$ for some $\psi\in\mathcal{L}$.

We are done if we can show that this is equivalent to m's being in the left-hand side which can be reformulated thus:

(ii) m satisfies all χ which are contained in $K\dot{-}(\phi\wedge\psi)$ for all $\psi\in\mathcal{L}$.

To see that *(i)* and *(ii)* are equivalent, we finally show that *(iii)* and *(iv)* are equivalent:

(iii) χ is satisfied by all m' that are in $[K\dot{-}(\phi\wedge\psi)]$ for some $\psi\in\mathcal{L}$.

(iv) χ is contained in $K\dot{-}(\phi\wedge\psi)$ for all $\psi\in\mathcal{L}$.

That *(iv)* entails *(iii)* is trivial for if χ is contained in all $K\dot{-}(\phi\wedge\psi)$, then all m' that are in some $[K\dot{-}(\phi\wedge\psi)]$ satisfy χ. To see that *(iii)* entails *(iv)*, suppose that *(iv)* is not true, i.e., that there is a ψ such that $\chi\notin K\dot{-}(\phi\wedge\psi)$. Then $K\dot{-}(\phi\wedge\psi)\cup\{\neg\chi\}$ is consistent, so there is an m'' such that $K\dot{-}(\phi\wedge\psi)\cup\{\neg\chi\}\subseteq m''$, which means that *(iii)* is not true.

In sum, then, we have shown that $\mathcal{S}(\mathcal{W}(\dot{-})) = (\mathcal{S}(\dot{-}))_{cl}$.

Whereas the construction just considered starts from an AGM contraction function $\dot{-}$, we might just as well start from a severe withdrawal function $\ddot{-}$, without changing the result. We know that $\dot{-} = \mathcal{C}(\ddot{-})$ is an AGM contraction, so by the result just proved, we get $(\mathcal{S}(\mathcal{C}(\ddot{-})))_{cl} = \mathcal{S}(\mathcal{W}(\mathcal{C}(\ddot{-})))$, but the latter is, by Observation 10*(ii)*, identical with $\mathcal{S}(\ddot{-})$. □

OBSERVATION 15. *(i) If $\mathcal{S}$ satisfies ($\mathcal{S}$1) – ($\mathcal{S}$4), then the function $\ddot{-}$ obtained from $\mathcal{S}$ by* (Def $\ddot{-}$ from $\mathcal{S}$) *is a severe withdrawal function.*
(ii) If $\ddot{-}$ is a severe withdrawal function, then $\ddot{-}$ can be represented as a sphere-based withdrawal, where the sphere system $\mathcal{S}$ on which $\ddot{-}$ is based is obtained by (Def $\mathcal{S}$ from $\dot{-}$) *(or equivalently, by* (Def $\mathcal{S}$ from $\ddot{-}$)*) and $\mathcal{S}$ satisfies ($\mathcal{S}$1) – ($\mathcal{S}$4).*

Proof. (i) Let $\mathcal{S}$ satisfy ($\mathcal{S}$1) – ($\mathcal{S}$4) and $\ddot{-}$ be obtained by (Def $\ddot{-}$ from $\mathcal{S}$). We need to show $\ddot{-}$ is a severe withdrawal function (i.e., satisfies ($\ddot{-}$1) – ($\ddot{-}$4), ($\ddot{-}$6), ($\ddot{-}$7a) and ($\ddot{-}$8)).

($\dot{-}1$) Directly by (Def $\dot{-}$ from $\mathcal{S}$) and definition of function th (see Section 3).

($\dot{-}2$) By (Def $\dot{-}$ from $\mathcal{S}$) we need to show $th(c_{\mathcal{S}}(\neg\phi)) \subseteq K$. Now, if $\vdash \phi$, then by (Def $c_{\mathcal{S}}$) we have $c_{\mathcal{S}}(\neg\phi) = [K]$. So Lemma 0*(i)* gives $th(c_{\mathcal{S}}(\neg\phi)) = th([K]) = K$ and the result holds trivially. Otherwise $\nvdash \phi$ and by ($\mathcal{S}2$) we have that $[K]$ is the $\subseteq$-minimum of $\mathcal{S}$ (i.e., $[K] \subseteq c_{\mathcal{S}}(\neg\phi)$). So by (Def $c_{\mathcal{S}}$) and Lemma 0*(iv)* the result is established.

($\dot{-}3$) Let $\phi \notin K$ or $\vdash \phi$. In the latter case, using (Def $c_{\mathcal{S}}$), (Def $\dot{-}$ from $\mathcal{S}$) and Lemma 0*(i)* we have that $K \dot{-} \phi = K$ and the result ensues directly. In the former case and supposing $\nvdash \phi$ we have that $[\neg\phi] \cap [K] \neq \emptyset$. Therefore ($\mathcal{S}2$) and (Def $c_{\mathcal{S}}$) give $c_{\mathcal{S}}(\neg\phi) = [K]$ and by (Def $\dot{-}$ from $\mathcal{S}$) and Lemma 0*(i)* we have $K \dot{-} \phi = K$ from which the desired result is obtained.

($\dot{-}4$) Let $\nvdash \phi$. By definition of $c_{\mathcal{S}}(\neg\phi)$, $[\neg\phi] \cap c_{\mathcal{S}}(\neg\phi) \neq \emptyset$. Therefore, using Lemma 0*(iv)* and (Def $\dot{-}$ from $\mathcal{S}$) we have $\phi \notin th(c_{\mathcal{S}}(\neg\phi)) = K \dot{-} \phi$.

($\dot{-}6$) Let $Cn(\phi) = Cn(\psi)$. Then $[\phi] = [\psi]$. If $\vdash \phi$, then $\vdash \psi$ and $K \dot{-} \phi = th([K]) = K \dot{-} \psi$. Otherwise $\nvdash \phi$ and $\nvdash \psi$. However $c_{\mathcal{S}}(\neg\phi) = c_{\mathcal{S}}(\neg\psi)$. Therefore $K \dot{-} \phi = th(c_{\mathcal{S}}(\neg\phi)) = th(c_{\mathcal{S}}(\neg\psi)) = K \dot{-} \psi$.

($\dot{-}7$a) Let $\nvdash \phi$. We need to show that $K \dot{-} \phi \subseteq K \dot{-} (\phi \wedge \psi)$. By (Def $\dot{-}$ from $\mathcal{S}$), we need to show that $th(c_{\mathcal{S}}(\neg\phi)) \subseteq th(c_{\mathcal{S}}(\neg(\phi \wedge \psi)))$. That is, by Lemma 0*(iv)*, $c_{\mathcal{S}}(\neg(\phi \wedge \psi)) \subseteq c_{\mathcal{S}}(\neg\phi)$ or, equivalently, $c_{\mathcal{S}}(\neg\phi \vee \neg\psi) \subseteq c_{\mathcal{S}}(\neg\phi)$. Since $\nvdash \phi$, then $\nvdash \phi \wedge \psi$. Now $[\neg\phi] \subseteq [\neg\phi \cup \neg\psi] = [\neg\phi] \cup [\neg\psi]$, so clearly $c_{\mathcal{S}}(\neg\phi \vee \neg\psi) \subseteq c_{\mathcal{S}}(\neg\phi)$ as desired.

($\dot{-}8$) Let $\phi \notin K \dot{-} (\phi \wedge \psi)$. By (Def $\dot{-}$ from $\mathcal{S}$), we have $K \dot{-} \phi = th(c_{\mathcal{S}}(\neg\phi)))$ and $K \dot{-} (\phi \wedge \psi) = th(c_{\mathcal{S}}(\neg(\phi \wedge \psi)))$. Since $\phi \notin K \dot{-} (\phi \wedge \psi) = th(c_{\mathcal{S}}(\neg(\phi \wedge \psi))) = th(c_{\mathcal{S}}(\neg\phi \vee \neg\psi))$, then $c_{\mathcal{S}}(\neg\phi \vee \neg\psi) \cap [\neg\phi] \neq \emptyset$. Therefore $c_{\mathcal{S}}(\neg\phi) \subseteq c_{\mathcal{S}}(\neg\phi \vee \neg\psi)$ and $th(c_{\mathcal{S}}(\neg\phi \vee \neg\psi)) \subseteq th(c_{\mathcal{S}}(\neg\phi))$ by Lemma 0*(iv)*. Thus $K \dot{-} (\phi \wedge \psi) \subseteq K \dot{-} \phi$ as desired.

(ii) Let $\dot{-}$ be a severe withdrawal function (i.e., satisfies ($\dot{-}1$) – ($\dot{-}4$), ($\dot{-}6$), ($\dot{-}7$a) and ($\dot{-}8$)) and let $\mathcal{S}$ be obtained from $\dot{-}$ by (Def $\mathcal{S}$ from $\dot{-}$) or, equivalently, (Def $\mathcal{S}$ from $\dot{-}$). We have to verify that (a) $\dot{-}'$ obtained from $\mathcal{S}$ using (Def $\dot{-}$ from $\mathcal{S}$) is identical to $\dot{-}$ and (b) that $\mathcal{S}$ satisfies the conditions for a system of spheres (i.e., ($\mathcal{S}1$) – ($\mathcal{S}4$)).

We prove (b) first as part of it will be useful in shortening the proof of (a).

(b) We verify that $\mathcal{S}$ is indeed a system of spheres centred on $[K]$.

($\mathcal{S}1$) The nestedness of spheres follows directly from (Def $\mathcal{S}$ from $\dot{-}$) and Lemma 2(*i*).

($\mathcal{S}2$) That $[K]$ is a sphere follows via (Def $\mathcal{S}$ from $\dot{-}$) and Lemma 0*(v)*) setting $\phi \equiv \top$ (or any $\psi \in \mathcal{L}$ such that $\vdash \psi$) since by ($\dot{-}3$) and ($\dot{-}2$) we have $K \dot{-} \top = K$. That $[K]$ is the $\subseteq$-minimal sphere then follows by (Def $\mathcal{S}$ from $\dot{-}$) using ($\dot{-}2$) and Lemma 0*(v)*).

($\mathcal{S}3$) That $\mathcal{M}_{\mathcal{L}}$ is a sphere follows directly by our constructions as we include it as a sphere.

($\mathcal{S}4$) Let $\nvdash \neg\phi$. We need to show that there is a sphere $U \in \mathcal{S}$ such that $U \cap [\phi] \neq \emptyset$ and $V \cap [\phi] \neq \emptyset$ implies $U \subseteq V$ for all $V \in \mathcal{S}$. We show that $U = [K\ddot{-}\neg\phi]$ satisfies this condition. Since $\nvdash \neg\phi$, then by ($\ddot{-}$4) $\neg\phi \notin K\ddot{-}\neg\phi$ so clearly $[K\ddot{-}\neg\phi] \cap [\phi] \neq \emptyset$. Now suppose for *reductio* there is some $V \in \mathcal{S}$ such that $V \cap [\phi] \neq \emptyset$ and $U \not\subseteq V$ (i.e., $V \subset U$ by ($\mathcal{S}1$) which has been shown above to hold). That is, by (Def $\mathcal{S}$ from $\ddot{-}$), there is some $\psi \in \mathcal{L}$ such that $[K\ddot{-}\psi] \cap [\phi] \neq \emptyset$ and $[K\ddot{-}\psi] \subset [K\ddot{-}\neg\phi]$. Since $[K\ddot{-}\psi] \cap [\phi] \neq \emptyset$, then $\neg\phi \notin K\ddot{-}\psi$. It follows by ($\ddot{-}$9) that $K\ddot{-}\psi \subseteq K\ddot{-}\neg\phi$ or, in other words, by Lemma 0*(v)*) $[K\ddot{-}\neg\phi] \subseteq [K\ddot{-}\psi]$ contradicting the above.

The proof of ($\mathcal{S}4$) actually shows that, for $\nvdash \neg\phi$, $c_{\mathcal{S}}(\phi) = [K\ddot{-}\neg\phi]$ (or, equivalently, that $\nvdash \phi$ implies $c_{\mathcal{S}}(\neg\phi) = [K\ddot{-}\phi]$) which can be conveniently used in the proof of (a).

(a) Whenever $\vdash \phi$, then $K\ddot{-}\phi = K$ by ($\ddot{-}$2) and ($\ddot{-}$3). Also, $c_{\mathcal{S}}(\neg\phi) = [K]$ by (Def $c_{\mathcal{S}}$) and consequently $K\ddot{-}'\phi = K$ by (Def $\ddot{-}$ from $\mathcal{S}$) and Lemma 0(*i*). Hence $K\ddot{-}\phi = K\ddot{-}'\phi = K$.

Consider, then, the case where $\nvdash \phi$.

Left to right. Suppose $\psi \in K\ddot{-}\phi$. We need to show $\psi \in K\ddot{-}'\phi$. Now clearly $[K\ddot{-}\phi] \subseteq [\psi]$. By following the proof of ($\mathcal{S}4$) we get $[K\ddot{-}\phi]$ is a sphere and $c_{\mathcal{S}}(\neg\phi) = [K\ddot{-}\phi]$. Therefore, $c_{\mathcal{S}}(\neg\phi) \subseteq [\psi]$. Hence $\psi \in th(c_{\mathcal{S}}(\neg\phi))$ and by Lemma 0*(iv)*) $\psi \in K\ddot{-}'\phi$ by (Def $\ddot{-}$ from $\mathcal{S}$) as desired.

Right to left. (The proof follows essentially be reversing that for the previous case.) Suppose $\psi \in K\ddot{-}'\phi$. We need to show $\psi \in K\ddot{-}\phi$. Since $\psi \in K\ddot{-}'\phi$, then $\psi \in th(c_{\mathcal{S}}(\neg\phi))$ by (Def $\ddot{-}$ from $\mathcal{S}$). Therefore, $c_{\mathcal{S}}(\neg\phi) \subseteq [\psi]$. Now $c_{\mathcal{S}}(\neg\phi) = [K\ddot{-}\phi]$ according to the proof of ($\mathcal{S}4$). Hence $[K\ddot{-}\phi] \subseteq [\psi]$ and thus $\psi \in K\ddot{-}\phi$ as desired. $\square$

LEMMA 16. *If* $\ddot{-}$ *is a severe withdrawal function, then the two conditions* (Def $\leq$ from $\dot{-}$) *and* (Def $\leq$ from $\ddot{-}$) *are equivalent.*

Proof. We have to show that

$$\phi \notin K\ddot{-}(\phi \wedge \psi) \quad \text{or} \quad \vdash (\phi \wedge \psi)$$

holds just in case

$$\phi \notin K\ddot{-}\psi \quad \text{or} \quad \vdash \psi$$

holds. To show that the former implies the latter, let $\phi \notin K\dot{-}(\phi \wedge \psi)$ or $\vdash (\phi \wedge \psi)$ and assume that $\nvdash \psi$. Hence $\nvdash (\phi \wedge \psi)$. Then $\phi \notin K\dot{-}(\phi \wedge \psi)$. Since $\nvdash \psi$, we get $K\ddot{-}\psi \subseteq K\ddot{-}(\phi \wedge \psi)$, by ($\ddot{-}$7a). So since $\phi \notin K\ddot{-}(\phi \wedge \psi)$, $\phi \notin K\ddot{-}\psi$, as desired.

For the converse, let $\phi \notin K\dot{-}\psi$ or $\vdash \psi$. From the former, we know that $\nvdash \phi$, by ($\ddot{-}$1). Now if $\vdash \psi$, then $\phi \notin K\ddot{-}\phi = K\ddot{-}(\phi \wedge \psi)$, by ($\ddot{-}$4) and ($\ddot{-}$6). So let $\nvdash \psi$ and thus $\phi \notin K\ddot{-}\psi$. Assume for reductio that $\phi \in K\ddot{-}(\phi \wedge \psi)$. Then by ($\ddot{-}$8c) $K\ddot{-}(\phi \wedge \psi) \subseteq K\ddot{-}\psi$. But then, since $\phi \notin K\ddot{-}\psi$, we get that $\phi \notin K\ddot{-}(\phi \wedge \psi)$, and we have a contradiction. Hence $\phi \notin K\ddot{-}(\phi \wedge \psi)$, as desired. $\square$

OBSERVATION 17. *Let $\dot{-}$ and $\ddot{-}$ be corresponding AGM contraction and severe withdrawal functions either via (Def $\ddot{-}$ from $\dot{-}$) or via (Def $\dot{-}$ from $\ddot{-}$). Then $\dot{-}$ and $\ddot{-}$ lead to identical entrenchment relations, via (Def $\leq$ from $\dot{-}$) and (Def $\leq$ from $\ddot{-}$).*

Proof. Let $\dot{-}$ and $\ddot{-}$ be corresponding AGM contraction and severe withdrawal functions either via (Def $\ddot{-}$from $\dot{-}$) or (Def $\dot{-}$from $\ddot{-}$). It follows from Observation 10 that it does not matter which of these definitions we apply.

Let $\leq$ be the epistemic entrenchment relation that arises from $\dot{-}$ via (Def $\leq$ from $\dot{-}$) and $\leq'$ be the epistemic entrenchment relation arising from $\ddot{-}$ via (Def $\leq$ from $\ddot{-}$).

We first show $\phi \leq \psi$ implies $\phi \leq' \psi$. Suppose $\phi \leq \psi$. Now $\phi \notin K\dot{-}(\phi \wedge \psi)$ or $\vdash \phi \wedge \psi$ by (Def $\leq$ from $\dot{-}$). In the former case (and assuming $\not\vdash \psi$ otherwise the result is trivial) $\phi \notin K\ddot{-}\psi$ by (Def $\ddot{-}$ from $\dot{-}$). In the latter case, surely $\vdash \psi$. In either case, $\phi \leq' \psi$ by (Def $\leq$ from $\ddot{-}$).

We now show $\phi \leq' \psi$ implies $\phi \leq \psi$. Suppose $\phi \leq' \psi$. Then $\phi \notin K\ddot{-}\psi$ or $\vdash \psi$ by (Def $\leq$ from $\ddot{-}$). If $\vdash \phi$, then the former case is not possible by ($\ddot{-}$1) and the latter case gives $\vdash \phi \wedge \psi$ whereby $\phi \leq \psi$ follows by (Def $\leq$ from $\dot{-}$).

Now let $\not\vdash \phi$. Consider first the case where $\phi \notin K\ddot{-}\psi$. By ($\ddot{-}$4) $\phi \notin K\ddot{-}\phi$. Now it follows by Lemma 2(ii) that $\phi \notin K\ddot{-}(\phi \wedge \psi)$. Equivalently, $\neg(\phi \wedge \psi) \rightarrow \phi \notin K\ddot{-}(\phi \wedge \psi)$ and consequently, by the Deduction Theorem, $\phi \notin Cn\,(K\ddot{-}(\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$. Therefore $\phi \notin K \cap Cn\,(K\ddot{-}(\phi \wedge \psi) \cup \{\neg(\phi \wedge \psi)\})$ and by (Def $\dot{-}$ from $\ddot{-}$) $\phi \notin K\dot{-}(\phi \wedge \psi)$ whereby $\phi \leq \psi$ follows by (Def $\leq$ from $\dot{-}$). Consider now the case where $\vdash \psi$. Then $\psi \in K\dot{-}(\phi \wedge \psi)$ by ($\dot{-}$1) which we know to hold by Observation 9 and therefore $\phi \notin K\dot{-}(\phi \wedge \psi)$ ($\dot{-}$4) (again, this holds by Observation 9). (Def $\leq$ from $\dot{-}$) now gives $\phi \leq \psi$ as desired. □

OBSERVATION 19. *(i) If $\leq$ satisfies (E1) – (E5), then the function $\ddot{-}$ obtained from $\leq$ by* (Def $\ddot{-}$ from $\leq$) *is a severe withdrawal function.*
(ii) If $\ddot{-}$ is a severe withdrawal function, then $\ddot{-}$ can be represented as an entrenchment-based withdrawal where the relation $\leq$ on which $\ddot{-}$ is based is obtained by (Def $\leq$ from $\dot{-}$) *(or equivalently, by* (Def $\leq$ from $\ddot{-}$)*), and $\leq$ satisfies (E1) – (E5).*

Proof.

(i) Assume that $\leq$ satisfies (E1) – (E5) and let $K\ddot{-}\phi = K \cap \{\psi : \phi < \psi\}$ when $\phi \in K$ and $\not\vdash \phi$, and $K\ddot{-}\phi = K$ otherwise. We have to verify that $\ddot{-}$ satisfies the postulates for severe withdrawals.

($\ddot{-}$1) Let $K\ddot{-}\phi \vdash \psi$. We want to show that $\psi \in K\ddot{-}\phi$. The case where $K\ddot{-}\phi = K$ is trivial, since K is a theory. So let $\phi \in K$ and $\not\vdash \phi$. By compactness, there are $\chi_1, \ldots, \chi_n \in K \ddot{-} \phi \subseteq K$ such that $\chi_1 \wedge \ldots \wedge \chi_n \vdash \psi$. Since K is a theory, $\chi_1 \wedge \ldots \wedge \chi_n$ and ψ are in K. So it remains to show that $\phi < \psi$. By repeated application of (E3), there is an i such that $\chi_i \leq \chi_1 \wedge \ldots \wedge \chi_n$. Since χ_i is in $K \ddot{-} \phi$, we have $\phi < \chi_i$. Hence, by the transitivity condition (E1), $\phi < \chi_1 \wedge \ldots \wedge \chi_n$. But $\chi_1 \wedge \ldots \wedge \chi_n \vdash \psi$, so by (E2), $\chi_1 \wedge \ldots \wedge \chi_n \leq \psi$. Hence, by (E1) again, $\phi < \psi$, so ψ is in $K \ddot{-} \phi$.

($\ddot{-}$2) and ($\ddot{-}$3) are immediate from (Def $\ddot{-}$ from $\leq$).

($\ddot{-}$4) Assume for reductio that $\nvdash \phi$ and $\phi \in K \ddot{-} \phi$. By the latter and (Def $\ddot{-}$ from $\leq$), we get $\phi \in K$. So by (Def $\ddot{-}$ from $\leq$) again, $\phi < \phi$, that is $\phi \leq \phi$ and $\phi \nleq \phi$ which is impossible.

($\ddot{-}$6) If $Cn(\phi) = Cn(\psi)$, then $\phi \in K$ iff $\psi \in K$, and $\nvdash \phi$ iff $\nvdash \psi$. It remains to show that $\phi < \chi$ iff $\psi < \chi$, for all χ. But this follows from $\phi \leq \psi$ and $\psi \leq \phi$, which is implied by (E2), and transitivity, (E1).

($\ddot{-}$7a) Let $\nvdash \phi$, and thus $\nvdash (\phi \wedge \psi)$. If $\phi \wedge \psi \notin K$, then $K \ddot{-} \phi \subseteq K = K \ddot{-} (\phi \wedge \psi)$ by (Def $\ddot{-}$ from $\leq$). If $\phi \wedge \psi \in K$, and thus $\phi \in K$, we need to show that $\phi < \chi$ implies $\phi \wedge \psi < \chi$ for all χ. But from (E2), we get $\phi \wedge \psi \leq \phi$, so the claim follows by transitivity, (E1).

($\ddot{-}$8) Let $\phi \notin K \ddot{-} (\phi \wedge \psi)$. Hence $\nvdash \phi$, by (E1), and also $\nvdash (\phi \wedge \psi)$. If $\phi \notin K$, so $K \ddot{-} (\phi \wedge \psi) \subseteq K = K \ddot{-} \phi$ by (Def $\ddot{-}$ from $\leq$). So let $\phi \in K$. Hence $K \ddot{-} (\phi \wedge \psi) \neq K$, so $\phi \wedge \psi \in K$. Hence $\phi \notin K \ddot{-} (\phi \wedge \psi)$ means that $\phi \wedge \psi \nless \phi$. Now assume that $\chi \in K \ddot{-} (\phi \wedge \psi)$, i.e., $\phi \wedge \psi < \chi$. We need to show that $\phi < \chi$. But since $\phi \wedge \psi \leq \phi$, by (E2), $\phi \wedge \psi \nless \phi$ means that $\phi \leq \phi \wedge \psi$. From this and $\phi \wedge \psi < \chi$, we get by transitivity (E1) that $\phi < \chi$ and therefore $\psi \in K \ddot{-} \phi$ by (Def $\ddot{-}$ from $\leq$), as desired.

(ii) Assume that $\ddot{-}$ satisfies ($\ddot{-}$1) – ($\ddot{-}$4), ($\ddot{-}$6), ($\ddot{-}$7a), ($\ddot{-}$8), and let $\phi \leq \psi$ if and only if $\phi \notin K \ddot{-} \psi$ or $\vdash \psi$. (That is, we use Lemma 16 and base the following on (Def $\leq$ from $\ddot{-}$) rather than directly on (Def $\leq$ from $\dot{-}$).) We have to verify (a) that the withdrawal function $\ddot{-}'$ obtained from $\leq$ with the help of (Def $\ddot{-}$ from $\leq$) is identical with $\ddot{-}$, and (b) that $\leq$ satisfies the defining conditions for epistemic entrenchment.

(a) Using the definition (Def $\ddot{-}$ from $\leq$) we get that $\psi \in K \ddot{-}' \phi$ iff

$$\psi \in K \text{ and } \begin{cases} \phi \notin K \text{ or } \vdash \phi & \text{or} \\ \phi < \psi \end{cases}$$

which means, by the definition (Def $\leq$ from $\ddot{-}$), that

$$(*) \qquad \psi \in K \text{ and } \begin{cases} \phi \notin K \text{ or } \vdash \phi & \text{or} \\ (\phi \notin K \ddot{-} \psi \text{ or } \vdash \psi) \text{ and } \psi \in K \ddot{-} \phi \text{ and } \nvdash \phi \end{cases}$$

First we show that $\psi \in K \ddot{-} \phi$ implies $\psi \in K \ddot{-}' \phi$. Suppose that $\psi \in K \ddot{-} \phi$ holds. If $\vdash \phi$ or $\phi \notin K$, then by ($\ddot{-}$3) and ($\ddot{-}$2) $K \ddot{-} \phi = K$, so $\psi \in K$ and $\psi \in K \ddot{-}' \phi$ by the upper line of $(*)$. So let $\nvdash \phi$ and $\phi \in K$. We have $\psi \in K \ddot{-} \phi \subseteq K$, by ($\ddot{-}$2). For the lower line of $(*)$, it remains to show that either $\vdash \psi$ or $\phi \notin K \ddot{-} \psi$. If $\nvdash \psi$, then, we need to show that $\phi \notin K \ddot{-} \psi$. But this follows from $\psi \in K \ddot{-} \phi$ by Lemma 2 *(iii)* (Expulsiveness).

For the converse, we show that $\psi \in K \ddot{-}' \phi$ implies $\psi \in K \ddot{-} \phi$. So let $(*)$ be given. From the upper line, we get $\psi \in K \ddot{-} \phi$ with the help of $(\ddot{-}3)$. So suppose that the lower line is true. But this line contains as a conjunct $\psi \in K \ddot{-} \phi$, which is just what we set out to prove.

(b) Finally we show that $\leq$ indeed satisfies (E1) – (E5).

(E1) Let $\phi \leq \psi$ and $\psi \leq \chi$, that is, $\phi \notin K \ddot{-} \psi$ or $\vdash \psi$, and also $\psi \notin K \ddot{-} \chi$ or $\vdash \chi$ by (Def $\leq$ from $\ddot{-}$). We need to show that $\phi \leq \chi$, i.e. $\phi \notin K \ddot{-} \chi$ or $\vdash \chi$. Assume that $\nvdash \chi$. Then $\psi \notin K \ddot{-} \chi$, and hence, by $(\ddot{-}1)$, $\nvdash \psi$. So we also have $\phi \notin K \ddot{-} \psi$. We conclude from $\psi \notin K \ddot{-} \chi$ with the help of $(\ddot{-}9)$ that $K \ddot{-} \chi \subseteq K \ddot{-} \psi$. Since $\phi \notin K \ddot{-} \psi$, we finally get $\phi \notin K \ddot{-} \chi$, as desired.

(E2) Let $\phi \vdash \psi$. In order to see that $\phi \leq \psi$, we need to show that $\phi \notin K \ddot{-} \psi$ or $\vdash \psi$. Assume $\nvdash \psi$. Then by $(\ddot{-}4)$ $\psi \notin K \ddot{-} \psi$. Hence by $(\ddot{-}1)$, $\phi \notin K \ddot{-} \psi$, as desired.

(E3) In order to see that either $\phi \leq \phi \wedge \psi$ or $\psi \leq \phi \wedge \psi$, we need to show that either $\phi \notin K \ddot{-} (\phi \wedge \psi)$ or $\vdash \phi \wedge \psi$, or $\psi \notin K \ddot{-} (\phi \wedge \psi)$ or $\vdash \phi \wedge \psi$. Assume that $\nvdash \phi \wedge \psi$. Then by $(\ddot{-}4)$, $\phi \wedge \psi \notin K \ddot{-} (\phi \wedge \psi)$, so by $(\ddot{-}1)$ in fact either $\phi \notin K \ddot{-} (\phi \wedge \psi)$ or $\psi \notin K \ddot{-} (\phi \wedge \psi)$.

(E4) Assume that $K \neq \mathcal{L}$. We need to show that $\phi \notin K$ just in case $\phi \leq \psi$ is true for every $\psi \in \mathcal{L}$. The latter condition means, by (Def $\leq$ from $\ddot{-}$), that

$$\phi \notin K \ddot{-} \psi \text{ or } \vdash \psi, \text{ for every } \psi \in \mathcal{L}$$

We know from $(\ddot{-}2)$ that $\phi \notin K$ is sufficient for this condition. To show that $\phi \notin K$ is also necessary, observe that the condition entails that $\phi \notin K \ddot{-} \bot$. Since $K \neq \mathcal{L}$, we know that $\bot \notin K$. So by $(\ddot{-}3)$, $K \ddot{-} \bot = K$. So $\phi \notin K$, as desired.

(E5) Assume that $\psi \leq \phi$ for all $\psi \in \mathcal{L}$. This means, by (Def $\leq$ from $\ddot{-}$), that either $\psi \notin K \ddot{-} \phi$ for all $\psi \in \mathcal{L}$ or $\vdash \phi$. The former cannot be, however, since if ψ is in *Cn* $(\emptyset)$, it will be in $K \ddot{-} \phi$ no matter what $K \ddot{-} \phi$ looks like, by $(\ddot{-}1)$. Hence $\vdash \phi$. □

LEMMA 20. *For any entrenchment relation $\leq$ with respect to K, the system of spheres $\mathcal{S}(\leq)$ satisfies conditions ($\mathcal{S}$1) – ($\mathcal{S}$4) with respect to $[K]$.*

Proof. Let $\leq$ be an entrenchment relation. We show that $\mathcal{S}(\leq)$ is indeed a system of spheres centred on $[K]$.

($\mathcal{S}$1) By the connectedness of $\leq$ (which follows from (E1)–(E3) — see [9, Lemma 3*(i)* p. 189]) we have that either $\phi \leq \psi$ or $\psi \leq \phi$ for $\phi,\ \psi \in \mathcal{L}$. It follows that either $\{\chi : \phi < \chi\} \subseteq \{\rho : \psi < \rho\}$ or $\{\rho : \psi < \rho\} \subseteq \{\chi : \phi < \chi\}$. Denoting $[\{\chi : \phi < \chi\}]$ by S_ϕ and $[\{\rho : \psi < \rho\}]$ by S_ψ as in (Def $\mathcal{S}$ from $\leq$), it follows by Lemma 0*(v)* and the fact the cuts are theories that $S_\phi \subseteq S_\psi$ or $S_\psi \subseteq S_\phi$ for $S_\phi,\ S_\psi \in \mathcal{S}(\leq)$.

($\mathcal{S}2$) If $K \neq \mathcal{L}$, then $\perp \notin K$ since K is a belief set. It follows by (E4) that $\{\phi : \perp < \phi\} = K$. Therefore $S_\perp = [\{\phi : \perp < \phi\}] = [K]$ and, by (Def $\mathcal{S}$ from $\leq$), $[K] \in \mathcal{S}(\leq)$. Now suppose for reductio that there is some $S_\psi \in \mathcal{S}(\leq)$ such that $S_\psi \subset [K]$. That is $[\{\chi : \psi < \chi\}] \subset [\{\phi : \perp < \phi\}]$. By Lemma 0*(iv)* and *(i)* (for cuts are theories — see [29, p. 159]) $\{\phi : \perp < \phi\} \subset \{\chi : \psi < \chi\}$. That is there is some $\rho \in \{\chi : \psi < \chi\}$ but $\rho \notin \{\phi : \perp < \phi\}$. Therefore $\psi < \rho$ but $\perp \not< \rho$. By the connectedness of $\leq$ we have $\rho \leq \perp$ and, by transitivity of $\leq$ (E1), $\psi < \perp$. This contradicts (E2) (for $\perp \vdash \psi$). Hence no such S_ϕ exists and $[K]$ is the $\subseteq$-minimum sphere of $\mathcal{S}(\leq)$. Otherwise $K = \mathcal{L}$ and $\emptyset \in \mathcal{S}$ by definition.

($\mathcal{S}3$) Take the cut $S_\top$. Now $\{\psi : \top < \psi\} = \emptyset$ by (E5). So $S_\top = [\{\psi : \top < \psi\}] = [\emptyset] = \mathcal{M}_\mathcal{L}$.

($\mathcal{S}4$) Let $\chi \in \mathcal{L}$ and $\nvdash \neg\chi$. It remains to show that $c_\mathcal{S}(\chi) = [\{\psi : \neg\chi < \psi\}]$ for all χ.

It follows from the compactness of *Cn* and from (E1) – (E3) that the set $\{\psi : \neg\chi < \psi\}$ does not entail $\neg\chi$ (see [29, proof of Lemma 5, p. 161]), so $[\{\psi : \neg\chi < \psi\}]$ intersects $[\chi]$. It remains to show that every S in $\mathcal{S}$ which is a proper subset of $[\{\psi : \neg\chi < \psi\}]$ does not intersect $[\chi]$. Suppose that S in $\mathcal{S}$ is a proper subset of $[\{\psi : \neg\chi < \psi\}]$. Then there is a ρ such that $S = [\{\psi : \rho < \psi\}]$ and $\{\psi : \rho < \psi\}$ is a proper superset of $\{\psi : \neg\chi < \psi\}$. Let ξ be in $\{\psi : \rho < \psi\} - \{\psi : \neg\chi < \psi\}$. Since $\xi \notin \{\psi : \neg\chi < \psi\}$ and $\leq$ is connected, $\xi \leq \neg\chi$. But then, since $\rho < \xi$, we can conclude with the help of (E1) that $\rho < \neg\chi$ as well. Thus $\neg\chi \in \{\psi : \rho < \psi\}$, so $[\{\psi : \rho < \psi\}]$ does not intersect $[\chi]$. We conclude that indeed $c_\mathcal{S}(\chi) = [\{\psi : \neg\chi < \psi\}]$. □

OBSERVATION 21. *For any entrenchment relation $\leq$, the AGM contractions and the severe withdrawals generated from $\leq$ and $\mathcal{S}(\leq)$ are identical, i.e., $\mathcal{C}(\mathcal{S}(\leq)) = \mathcal{C}(\leq)$ and $\mathcal{W}(\mathcal{S}(\leq)) = \mathcal{W}(\leq)$.*

Proof. Let $\leq$ be an entrenchment relation. We know from Lemma 20 that $\mathcal{S}(\leq)$ is indeed a system of spheres centred on $[K]$.

It remains to show that for $\mathcal{S} = \mathcal{S}(\leq)$ generated by (Def $\mathcal{S}$ from $\leq$) it holds that

(i) $\{\psi \in K : \phi < \phi \vee \psi\} = th([K] \cup f_\mathcal{S}(\neg\phi))$

(ii) $\{\psi \in K : \phi < \psi\} = th(c_\mathcal{S}(\neg\phi))$

We begin by showing *(ii)*: Using the proof of the Lemma 20, the part concerning $\mathcal{S}4$, we know that for every ϕ, the smallest sphere intersecting $[\neg\phi]$, that is $c_\mathcal{S}(\neg\phi)$, is identical with the set $[\{\psi : \phi < \psi\}]$. Moreover, since $\{\psi : \phi < \psi\}$ is a theory (see [29, p. 159]), by Lemma 0*(i)*, $th(c_\mathcal{S}(\neg\phi) = \{\psi : \phi < \psi\}$. Furthermore, we conclude $f_\mathcal{S}(\neg\phi) = c_\mathcal{S}(\neg\phi) \cap [\neg\phi] = [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]$.

Knowing this, we have to show for *(i)* that

$$th([K] \cup [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]) = \{\psi \in K : \phi < \phi \vee \psi\}$$

To show that the left-hand-side is included in the right-hand-side, let $\chi \in m$ for all $m \in [K] \cup [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]$. We need to show that $\chi \in K$ and $\phi < \phi \vee \chi$. But we know from the assumption that $\chi \in m$ for all $m \in [K]$. Since K is a theory, $\chi \in K$. Moreover, $\chi \in m$ for all $m \in [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]$. Hence $\phi \vee \chi \in m$ for all $m \in [\{\psi : \phi < \psi\}]$. Hence, by the completeness of *Cn*, $\{\psi : \phi < \psi\} \vdash \phi \vee \chi$. Since $\{\psi : \phi < \psi\}$ is a theory (see [29, p. 159]), we get that $\phi \vee \chi \in \{\psi : \phi < \psi\}$, that is, $\phi < \phi \vee \chi$.

To show conversely that the right-hand-side is included in the left-hand-side, let $\chi \in K$ and $\phi < \phi \vee \chi$. Clearly $\chi \in m$ for all $m \in [K]$, since $\chi \in K$. It remains to show that $\chi \in m$ for all $m \in [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]$. Let such an m be given. Since $\phi < \phi \vee \chi$, we know that from $m \in [\{\psi : \phi < \psi\}]$ we can infer that $\phi \vee \chi \in m$. But also $\neg\phi \in m$ (since $\phi \notin \{\psi : \phi < \psi\}$ — see [29, proof of Lemma 5, p. 161]). So, since m is a theory, $\chi \in m$. We have shown that $\chi \in m$ for all m in $[K] \cup [\{\psi : \phi < \psi\} \cup \{\neg\phi\}]$. But by the definition of th, this just means that χ is in $th([K] \cup [\{\psi : \phi < \psi\} \cup \{\neg\phi\}])$, as desired. $\square$

LEMMA 22. *For any system of spheres $\mathcal{S}$ with respect to $[K]$, the entrenchment relation $\mathcal{E}(\mathcal{S})$ satisfies conditions (E1) – (E5) with respect to K.*

Proof. Let $\mathcal{S}$ be a system of spheres centred on $[K]$. We show that $\leq$ is an epistemic entrenchment relation on K.

(E1) Let $\phi \leq \psi$ and $\psi \leq \chi$. By (Def$'$ $\leq$ from $\mathcal{S}$) we have that $c_{\mathcal{S}}(\neg\psi) \not\subseteq [\phi]$ and $c_{\mathcal{S}}(\neg\chi) \not\subseteq [\psi]$. It follows that $c_{\mathcal{S}}(\neg\psi) \cap [\neg\phi] \neq \emptyset$ and $c_{\mathcal{S}}(\neg\chi) \cap [\neg\psi] \neq \emptyset$. Consequently, from the latter, $c_{\mathcal{S}}(\neg\psi) \subseteq c_{\mathcal{S}}(\neg\chi)$. Therefore $c_{\mathcal{S}}(\neg\chi) \cap [\neg\phi] \neq \emptyset$. Hence $c_{\mathcal{S}}(\neg\chi) \not\subseteq [\phi]$ and, by (Def$'$ $\leq$ from $\mathcal{S}$) $\phi \leq \chi$ as desired.

(E2) Let $\phi \vdash \psi$. It follows that $[\phi] \subseteq [\psi]$. Suppose for *reductio* that $c_{\mathcal{S}}(\neg\psi) \subseteq [\phi]$. It follows that $c_{\mathcal{S}}(\neg\psi) \subseteq [\phi]$ which contradicts the definition of $c_{\mathcal{S}}(\neg\psi)$. Hence $c_{\mathcal{S}}(\neg\psi) \not\subseteq [\phi]$ as required.

(E3) Suppose $\phi \not\leq \phi \wedge \psi$. We need to show that $\psi \leq \phi \wedge \psi$. From our supposition an (Def$'$ $\leq$ from $\mathcal{S}$) it follows that $c_{\mathcal{S}}(\neg(\phi \wedge \psi)) \subseteq [\phi]$. Now $[\neg(\phi \wedge \psi)] = [\neg\phi \vee \neg\psi] = [\neg\phi] \cup [\neg\psi]$. Therefore $c_{\mathcal{S}}(\neg(\phi \wedge \psi)) \cap [\neg\psi] \neq \emptyset$. Consequently $c_{\mathcal{S}}(\neg(\phi \wedge \psi)) \not\subseteq [\psi]$ and hence by (Def$'$ $\leq$ from $\mathcal{S}$) we have $\psi \leq \phi \wedge \psi$ as desired.

(E4) Let $K \neq \mathcal{L}$ We need to show $\phi \leq \psi$ for every $\psi \in \mathcal{L}$ iff $\phi \notin K$.

Left to right. we shall prove the contrapositive. Let $\phi \in K$. We need to show $\phi \not\leq \psi$ for some $\psi \in \mathcal{L}$. By (Def$'$ $\leq$ from $\mathcal{S}$) this amounts to showing $c_{\mathcal{S}}(\neg\psi) \subseteq [\phi]$ for some $\psi \in \mathcal{L}$. Consider $\psi \equiv \neg\psi$. $c_{\mathcal{S}}(\neg\neg\phi) = c_{\mathcal{S}}(\phi) = [K]$

by ($\mathcal{S}2$) and since $\phi \in K$. Since $\phi \in K$ it follows that $[K] \subseteq [\phi]$ and therefore $c_{\mathcal{S}}(\phi) \subseteq [\phi]$ as desired.

Right to left. We shall prove the contrapositive. Let $\phi \leq \psi$ for some $\psi \in \mathcal{L}$. By (Def$'$ $\leq$ from $\mathcal{S}$), there is some $\psi \in \mathcal{L}$ such that $c_{\mathcal{S}}(\neg\psi) \subseteq [\phi]$. Consequently by ($\mathcal{S}2$), $[K] \subseteq c_{\mathcal{S}}(\neg\psi) \subseteq [\phi]$. Hence $\phi \in K$ as required.

(E5) Let $\not\vdash \phi$. Again, we show this by considering the contrapositive. We need to show that $\psi \not\leq \phi$ for some $\psi \in \mathcal{S}$. That is, by (Def$'$ $\leq$ from $\mathcal{S}$), we need to show $c_{\mathcal{S}}(\neg\phi) \subseteq [\psi]$ for some $\psi \in \mathcal{L}$. take $\psi \equiv \top$. Clearly $c_{\mathcal{S}}(\neg\phi) \subseteq [\top]$ as desired. □

OBSERVATION 23. *Let $\leq$ be an entrenchment relation and $\mathcal{S}$ a system of spheres. Then*

(i) $\mathcal{E}(\mathcal{S}(\leq)) = \leq$.

(ii) $\mathcal{S}(\mathcal{E}(\mathcal{S}))$ is the topological closure of the trimming of $\mathcal{S}$, i.e., $(\mathcal{S}_t)_{cl}$.

Proof. *(i)* Let $\leq$ be an entrenchment relation.

Furthermore, let $\leq'$ be $\mathcal{E}(\mathcal{S}(\leq))$. That $\mathcal{S}(\leq)$ is a system of spheres was proved in Observation 21*(i)* and that $\leq' = \mathcal{E}(\mathcal{S}(\leq))$ is an epistemic entrenchment relation subsequently follows by the proof of Observation 24 below.

Now $\phi \leq' \psi$ iff $c_{\mathcal{S}}(\neg\psi) \not\subseteq [\phi]$ iff by (Def$'$ $\leq$ from $\mathcal{S}$). This is the case iff $[\{\chi : \psi < \chi\}] \not\subseteq [\phi]$ by (Def $\mathcal{S}$ from $\leq$) (see beginning of the proof of Observation 21*(i)*) which holds iff $\{\chi : \psi < \chi\} \not\vdash \phi$. This holds iff $\psi \not< \phi$ iff (since $\{\chi : \psi < \chi\}$ is a theory (see [29, proof of Lemma 5, p. 161])) which holds iff $\phi \leq \psi$ by the connectedness of $\leq$ which follows from (E1)–(E3) (see [9, Lemma 3*(i)* p. 189]) as desired.

(ii) Let $\mathcal{S}$ be a system of spheres. Furthermore, let $\mathcal{S}'$ be $\mathcal{S}(\mathcal{E}(\mathcal{S}))$. That $\mathcal{E}(\mathcal{S})$ is indeed an epistemic entrenchment relation has been shown in Lemma 22 and that $\mathcal{S}(\mathcal{E}(\mathcal{S}))$ is a system of spheres follows by the proof of Observation 21*(i)*.

Now S is in $\mathcal{S}'$ iff there is some ϕ such that $S = [\{\psi : \phi < \psi\}]$, by (Def $\mathcal{S}$ from $\leq$). This holds, by (Def$'$ $\leq$ from $\mathcal{S}$), iff there is some ϕ such that $S = [\{\psi : c_{\mathcal{S}}(\neg\phi) \subseteq [\psi]\}]$. Now this holds iff there is some ϕ such that $S = [\{\psi : \psi \in th(c_{\mathcal{S}}(\neg\phi))\}]$, Simplifying, this is the case iff there is some ϕ such that $S = [th(c_{\mathcal{S}}(\neg\phi))]$ as desired.

Now the trimming of $\mathcal{S}$ consists exactly of all the sets of the form $c_{\mathcal{S}}(\chi)$, and the operation of taking the topological closure is just the one that takes every set X of worlds to $[th(X)]$. Thus we have proved that $\mathcal{S}'$ is the topological closure of the trimming of $\mathcal{S}$. □

OBSERVATION 24. *For any system of spheres $\mathcal{S}$, the AGM contractions and the severe withdrawals generated from $\mathcal{S}$ and $\mathcal{E}(\mathcal{S})$ are identical, i.e., $\mathcal{C}(\mathcal{E}(\mathcal{S})) = \mathcal{C}(\mathcal{S})$ and $\mathcal{W}(\mathcal{E}(\mathcal{S})) = \mathcal{W}(\mathcal{S})$.*

Proof. Let $\mathcal{S}$ be a system of spheres centred on $[K]$, and $\leq = \mathcal{E}(\mathcal{S})$ as defined by (Def$'$ $\leq$ from $\mathcal{S}$). Notice that since $\leq$ is connected (see [9, Lemma 3*(i)* p. 189]), we have that $\phi < \psi$ if and only if $c_{\mathcal{S}}(\neg\phi) \subseteq [\psi]$.

Lemma 22 showed that $\leq$ is indeed an epistemic entrenchment relation with respect to K.

For the limiting case where $\vdash \phi$ all contractions and withdrawals with respect to ϕ are set to K, so we assume in the following that $\nvdash \phi$.

Since $\leq = \mathcal{E}(\mathcal{S})$ is an entrenchment relation, we know from Observation 21 that $\mathcal{C}(\mathcal{E}(\mathcal{S})) = \mathcal{C}(\mathcal{S}(\mathcal{E}(\mathcal{S})))$. But since by Observation 23, $\mathcal{S}(\mathcal{E}(\mathcal{S})) = (\mathcal{S}_t)_{cl}$, we conclude with Lemma 12 that $\mathcal{C}(\mathcal{E}(\mathcal{S})) = \mathcal{C}((\mathcal{S}_t)_{cl}) = \mathcal{C}(\mathcal{S})$.

Precisely the same argument shows that $\mathcal{W}(\mathcal{E}(\mathcal{S})) = \mathcal{W}(\mathcal{S})$. □

B. Twelve Methods of Withdrawing a Belief ϕ

In this appendix we contrast various methods for withdrawal of a belief ϕ from a belief set K currently found in the literature. We consider the principal case where $\nvdash \phi$ as the majority of these methods satisfy the failure property (i.e., $\vdash \phi$ implies $K \dot{-} \phi = K$).

The following table lists a number of proposals together with an indication of which $\neg\phi$-worlds and which ϕ-worlds are contained in $[K \dot{-} \phi]$. S_ϕ refers to the smallest sphere intersecting ϕ (i.e., $c_{\mathcal{S}}(\phi)$). $[K]$ is, of course, the smallest (innermost) sphere.

	$[K \dot{-} \phi]$	... within $[\neg\phi]$	... within $[\phi]$
1.	AGM (trans. rel.) partial meet [1]	$[\neg\phi] \cap S_\phi$	$[K]$
2.	Severe withdrawal [Section 3]	$[\neg\phi] \cap S_\phi$	$[\phi] \cap S_\phi$
3.	AGM maxichoice [1]	single $\neg\phi$-world	$[K]$
4.	Saturatable set [18, 14])	single $\neg\phi$-world	some X s.th. $[K] \subseteq X \subseteq [\phi]$
5.	Partial meet of saturatable sets [14]	$[\neg\phi] \cap S_\phi$	some X s.th. $[K] \subseteq X \subseteq [\phi]$
6.	Iron-fisted withdrawal [Section 7]	$[\neg\phi] \cap S_\phi$	$[\phi]$
7.	Levi – damped type 1 [20]	$[\neg\phi] \cap S_\phi$	$[\phi] \cap S_2$ [28]
8.	Cantwell fallback-based [4]	$[\neg\phi] \cap S_\phi$	$[\phi] \cap S_i$ for some $i \in \{1, \ldots, n\}$
9.	Systematic withdrawal [26]	$[\neg\phi] \cap S_\phi$	$[\phi] \cap S_{\phi-1}$
10.	Lindström and Rabinowicz [22]	$[\neg\phi] \cap S_\phi$	some X s.th. $[K] \subseteq X \subseteq [\phi] \cap S_\phi$
11.	Semi-contraction [6]	$[\neg\phi] \cap S_\phi$	some X s.th. $[K] \subseteq X \subseteq [\phi] \cap S_\phi$ [29]
12.	Nayak [p.c.]	$[\neg\phi] \cap S_\phi$	$[\phi] \cap (S_\phi - S_{\phi-1})$ [30]

The diagrams on the following pages illustrate typical situations for these various proposals. Note that Figures 1 and 2 appeared in Section 3.

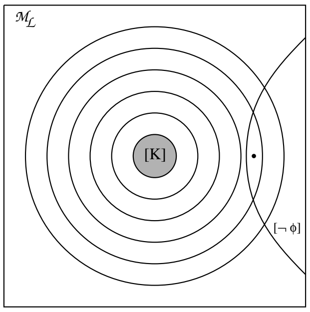


Figure 3. Maxichoice — pure minimal change (wrt ⊆).

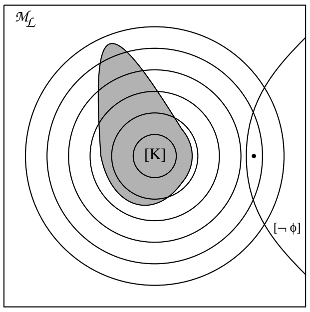


Figure 4. Saturatable set (no recovery).

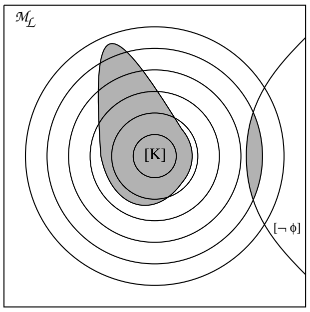


Figure 5. Partial meet of saturatable sets.

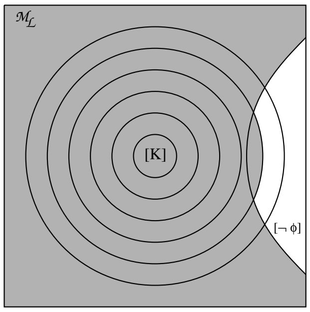


Figure 6. "Iron-fisted" withdrawal — minimal revision equivalent withdrawal

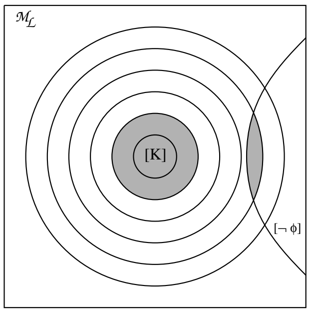


Figure 7. Levi Contraction via damped informational value of type 1.

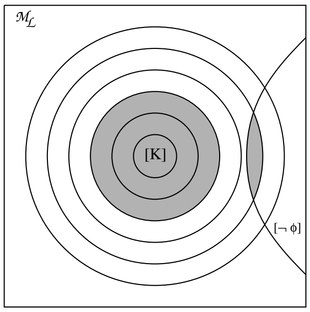


Figure 8. Cantwell "fallback-based".

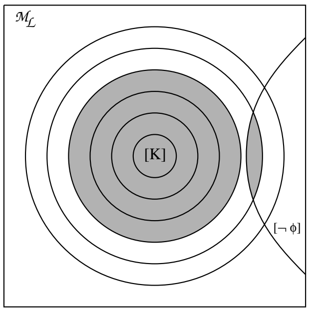


Figure 9. Meyer *et al.* systematic withdrawal.

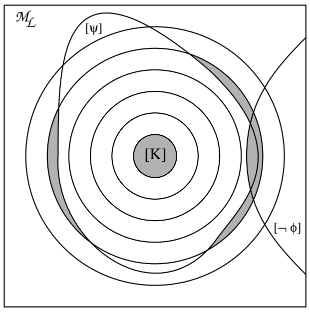


Figure 11. Fermé and Rodriguez semi-contraction.

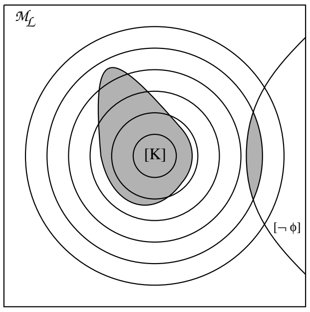


Figure 10. Lindström and Rabinowicz (Interpolation).

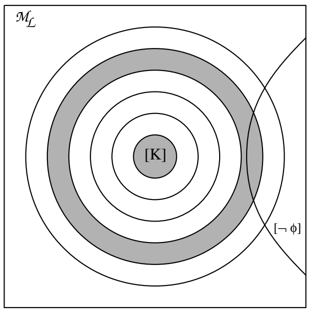
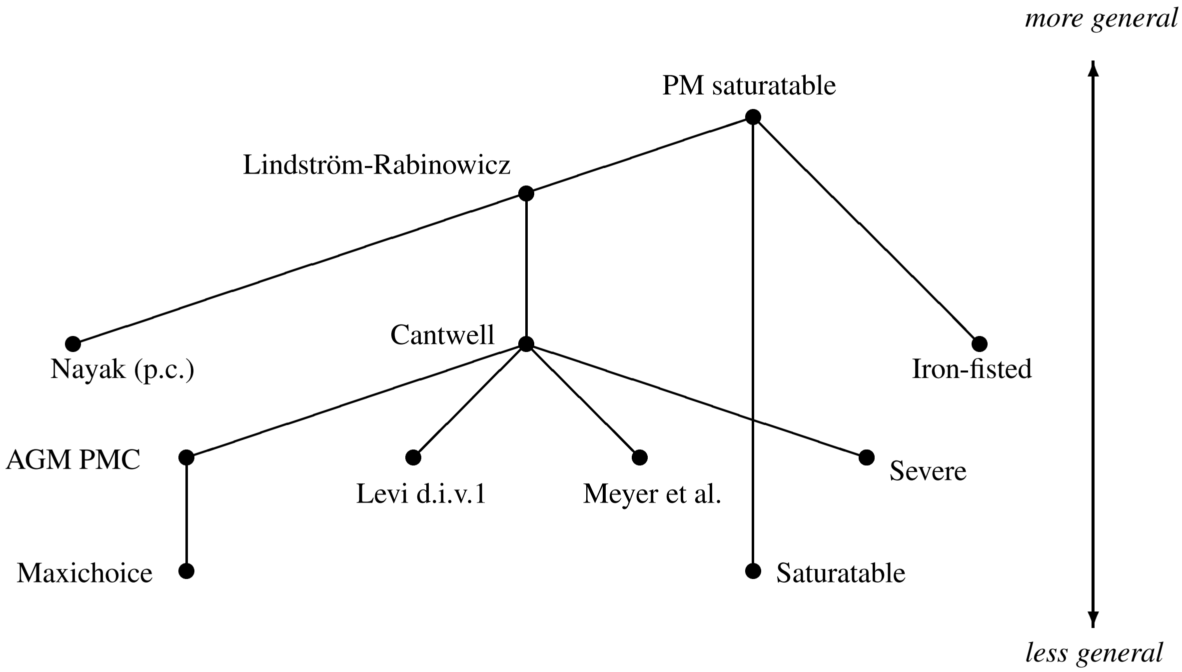


Figure 12. Nayak (personal communication).

C. Interrelationship Between Methods



Notes

[1] In the AGM literature these two terms have a precise meaning, differentiating important forms of belief removal, which we shall introduce later. For the time being, however, we defer to the term contraction when referring to any operation removing beliefs from an agent's epistemic state.

[2] Levi [18] refers to this as *coerced contraction* (as distinct from *uncoerced contraction* which refers to belief removal for purposes described in what follows).

[3] Even more so if one considers the rather trivial nature of AGM expansion.

[4] This generalisation allows us to retain the spirit of the original while not being tied down to loaded terms such as 'information'.

[5] A contraction of K by $\phi \in \mathcal{L}$ in the AGM framework is *maxichoice* if it leads to a maximal subset of K that does not imply ϕ. It is *partial meet* if it is the intersection of select maxichoice contractions.

[6] A revision function defined from a maxichoice AGM contraction function via the Levi Identity [8, p. 69] ($K * \phi = Cn(K \dot{-} (\neg\phi) \cup \{\phi\})$) always returns a belief state which is maximally consistent and therefore has an opinion as to the truth or falsity of every sentence in the object language.

[7] More formally, $K' \in K \bot \phi$ if and only if *(i)* $K' \subseteq K$; *(ii)* $\phi \notin Cn(K')$; and, *(iii)* for any K'' such that $K' \subset K'' \subseteq K$, $\phi \in Cn(K'')$.

[8] An AGM full meet contraction may be constructed from K as $K \dot{-} \phi = \bigcap (K \bot \phi)$.

[9] Strictly speaking, Grove's [12] construction deals solely with syntactic, rather than semantic, objects; maximally consistent sets of sentences take the place of worlds. It can be thought of as furnishing a semantics in so far as it provides a "picture" for the belief change process.

[10] In traditional AGM terminology, $K \dot{-} \phi \in K \bot \phi$ where $K \bot \phi$ is the set of maximal subsets of K failing to imply ϕ. The connection has been established by Grove [12].

[11] Recently, Fermé and Rodriguez [7] have also, independently, proposed an axiomatisation for severe withdrawal using the postulate ($\ddot{-}9$) — see Section 6.

[12] Technically, this can be viewed as the dual of the Grove ordering. See Gärdenfors [8, Section 4.8].

[13] It is easy to show that a necessary and sufficient condition for (Def $\dot{-}$ from $\leq$) to generate maxichoice contractions is that the entrenchment relation satisfies either $\phi < \psi$ or $\psi < (\psi \rightarrow \phi)$, for all sentences ϕ and ψ in $\mathcal{L}$. See Rott [31, Chapter 8].

[14] Or among those for nonmonotonic consequence relations. A nonmonotonic consequence relation $\mid\!\sim$ can be defined from a revision operator $*$ and a belief set (or rather, expectation set) K by putting $\phi \mid\!\sim \psi$ iff $\psi \in K * \phi$ [10]. The parameter K is here left implicit.

[15] However, via the so-called Harper Identity — $K \dot{-} \phi = K \cap K * \neg\phi$ — Recovery can be linked with success for revision operations and reflexivity for nonmonotonic consequence relations.

[16] Cut: If $\phi \mid\!\sim \psi$ and $\phi \wedge \psi \mid\!\sim \chi$, then $\phi \mid\!\sim \chi$.
Cumulative Monotony: If $\phi \mid\!\sim \psi$ and $\phi \mid\!\sim \chi$, then $\phi \wedge \psi \mid\!\sim \chi$.
Or: If $\phi \mid\!\sim \chi$ and $\psi \mid\!\sim \chi$, then $\phi \vee \psi \mid\!\sim \chi$.
Rational Monotony: If $\phi \not\mid\!\sim \neg\psi$ and $\phi \mid\!\sim \chi$, then $\phi \wedge \psi \mid\!\sim \chi$.
For a detailed discussion, we refer the reader to Makinson's [24] survey.

[17] Rott [31] considers a slightly different set of postulates in an attempt to remove reference to the underlying logic. We shall remain closer in spirit to the AGM as this additional generality does not affect our aims in this paper.

[18] We are grateful to Sven Ove Hansson for highlighting this property.

[19] In fact, Makinson's [23] result regarding the maximality of AGM contraction functions bears this out also.

[20] That is, X is in $\mathcal{S}$ iff it is a fixed point in the operation taking every model set Y to $\bigcup \{[K \dot{-} \psi] : Y \not\subseteq [\psi]\}$

[21] For some purposes it is convenient to rephrase this definition in terms of contractions of conjuncts as follows.

$$\text{(Def}' \ \mathcal{S} \text{ from } \dot{-}) \qquad X_\phi = \begin{cases} \bigcup\{[K \dot{-} \phi \wedge \psi] : \psi \in \mathcal{L}\} & \text{whenever } \not\vdash \phi \\ [K] & \text{otherwise} \end{cases}$$

[22] In model theoretic terms, the spheres of the Lewis-Grove constructions are $\Delta\Sigma$-elementary but not Δ-elementary (see [3, p. 141]). If, however, their second construction is applied to severe withdrawals, the resulting spheres turn out to be Δ-elementary.

[23] Similarly all constructions of systems of spheres from some entrenchment relation $\leq$ (which will use "cuts" or "up-sets" with respect to $\leq$, see Section 12 below) yield Δ-elementary spheres.

[24] Priest *et al.* [28] have pointed to an error in Grove's [12, Theorem 1] proof verifying that the revision postulate analogues of ($\dot{-}7$) and ($\dot{-}8$) are satisfied by a revision function derived

from a system of spheres. They demonstrate one way to fix Grove's proof. Alternatively, they suggest that every sphere in $\mathcal{M}_{\mathcal{L}}$ be required to be elementary. In any case, Grove's statement of the result is not at fault.

[25] A generalisation of this theorem for relations of epistemic entrenchment with incomparabilities (and ones that need not satisfy Minimality and Maximality) is given in Rott [30, Theorem 2].

[26] Under the same assumptions it also reduces to what might be called the standard definition in the literature (cf. Gärdenfors [8, pp. 95–96]):

(Def$''$ $\leq$ from $\mathcal{S}$) $\quad \phi \leq \psi$ iff $c_{\mathcal{S}}(\neg\phi) \subseteq c_{\mathcal{S}}(\neg\psi)$.

[27] We change the notation of Kaluzhny and Lehmann in order to avoid confusion with the notation used in this paper.

[30] S_2 refers to the second smallest sphere, that is, that sphere X such that $[K] \subset X$ and $X \subseteq Y$ for all $Y \neq [K]$.

[30] $S_{\phi-1}$ refers to the sphere immediately smaller than S_ϕ, that is, that sphere X such that $X \subset S_\phi$ and $Y \subseteq X$ for all $Y \subset S_\phi$.

[30] This is for reasonable semi-contraction functions (see [6, Section 5]). Otherwise we have, within $[\phi]$, some X s.th. $[K] \subseteq X \subseteq \mathcal{M}_{\mathcal{L}}$.

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