On the Mixing Rate of the Triangulation Walk*

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Abstract

Let T_n denote the set of triangulations of a convex polygon K with n sides. We study the random walk on T_n whose transitions are "flips" of one of the $n-3$ internal diagonals of the current triangulation, the choice of diagonal being random. By bounding the conductance of this graph we show that the walk mixes rapidly, namely in time $O(n^\beta)$. A direct argument is given for the fact that the mixing rate is at least $\Omega(n^{3/2})$.

1 Introduction and Summary

Consider a convex polygon K on n points $p_0, \ldots, p_{n-1}$, in clockwise order. A triangulation is a set of $n-3$ non-crossing diagonals p_ip_j which partitions K into $n-2$ triangles. It is familiar (see e.g., [7]) that T_n, the set of all triangulations of K, satisfies

$$|T_n| = \gamma_{n-2} \equiv \frac{1}{n-1}\binom{2n-4}{n-2}, \tag{1}$$

the right hand side being the $(n-2)^{nd}$ Catalan number. We consider the graph G_n whose vertex set is $V = T_n$. Edges are triangulations which differ

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by one diagonal. Specifically, a diagonal of a triangulation $\tau \in V$ may be "flipped"; i.e., removed - creating a quadrilateral - and replaced in the opposite way to give a new triangulation τ' adjacent to τ in G_n. Thus G_n is $n-3$ regular.

It is natural to consider the random walk on G_n which moves to each neighbor of a vertex with equal chance. The key question is how quickly the walk converges to the steady-state distribution, in this case, the uniform. Often, there are practical reasons for studying such a random walk on a set S of combinatorial objects, for example to estimate the size of S or to generate an element of S at random. However we already know $|G_n| = \gamma_{n-2}$ and there already exist optimal algorithms to generate random triangulations [3], [1], [2]. But there are interesting sets which are related to T_n, e.g., $T_n(P)$, the set of triangulations of the non-convex, simple polygon P, and $T_n(K)$, the triangulations of the set K whose points are *not* in convex position; in both these cases the task of computing the size, and of generating a random element are open problems. By studying the walk on T_n we may learn how to study other sets of triangulations.

Unfortunately little is known about the triangulation walk. This is surprising because G_n is so familiar and well-studied [7], [12] (also [4], [5], [6], [8], [10], which are relevant because G_n is also the graph of rooted binary trees with $n-2$ internal nodes, adjacency defined by the rotation operation). In the present paper we will apply the method of conductance to show that the walk mixes rapidly. Write $\overline{S}$ for $T_n \backslash S$. We prove

Theorem 1 *For any set $S \subset T_n$, there is a matching from S to $\overline{S}$ with*

$$\geq \frac{|S||\overline{S}|}{|T_n|n^\beta}$$

edges of G_n.

This would imply that the conductance is greater than $1/(2n^{\beta+1})$ and the mixing rate $O(n^{2\beta+3})$ [11]. We will prove this with $\beta = 11$ but think we can reduce it to 4 or 5. In addition we show

Theorem 2 *The mixing rate is at least $\Omega(n^{3/2})$.*

It is interesting here to compare this to the walk on the n-cube (similar to G_n, it is an n-regular graph with c^n vertices), which mixes in $O(n \log n)$ steps.

A key idea behind these results is a partitioning of T_n into sets in which all triangulations share a certain "central triangle" t. This allows us to apply induction on the 3 subpolygons defined by t. It also gives us the ability to move between sets of triangulations in a systematic way and to construct a matching with many edges from S to $\overline{S}$.

In the next section we collect some notations, definitions, and useful facts about T_n. The lower bound is proved in Section 3. We do this first because it is much more pleasant and easier than the proof of the conductance bound, which is described in Section 4.

2 Some Preliminaries

Again, G_n is the graph with vertex set $V = T_n$, the set of triangulations of the convex polygon K whose vertices are the points $p_0, \ldots, p_{n-1}$, written in clockwise order; triangulations τ_1 and τ_2 in V are adjacent if τ_2 may be obtained by flipping one of the $n-3$ diagonals of τ_1. The triangulation walk moves from a vertex τ to one of its $n-3$ neighbors, each with equal probability. This gives an ergodic Markov chain $MC(n)$ which is reversible, and its stationary distribution is the uniform probability on V.

If p_ip_j is a diagonal of a triangulation $\tau \in V$, $i > j$, define its length to be $\|p_ip_j\| = \min(i-j, n-i+j)$, the (fewest) number of successive edges of K between the endpoints. Every triangulation has a longest diagonal ℓ and it is clear that $n/3 \leq \|\ell\| \leq n/2$, the upper bound attainable only if n is even. The longest diagonal must be in a triangle $t = \Delta p_ip_jp_k$, which has the sum of the lengths of its three diagonals equal to n. We call t a *central triangle* and observe that it is unique unless n is even and τ has a diagonal ℓ with length $n/2$. In this case there are *two* such triangles, and to clarify the situation, we call $\ell = p_ip_{i+n/2}$ a *central edge* and regard it as a *degenerate central triangle.* The diagonals comprising a nondegenerate central triangle $t = \Delta p_ip_jp_k$ divide K into three subpolygons of sizes $\|p_ip_j\|+1$, $\|p_jp_k\|+1$, and $\|p_kp_i\|+1$. Write

$$C_n = \{t = \Delta p_ip_jp_k : t \text{ a central triangle}\}$$

and observe that $|C_n| = \Theta(n^3)$. In fact $|C_n| \leq n^3/12$ for $n \geq 10$.

It will be useful to regard central triangles as vertices in a graph, $G(C_n)$. A non-degenerate central triangle $t = \Delta p_ip_jp_k$ is adjacent to another non-degenerate central triangle t' if t and t' have two points of K in common and the third points differ by exactly one. A degenerate central triangle

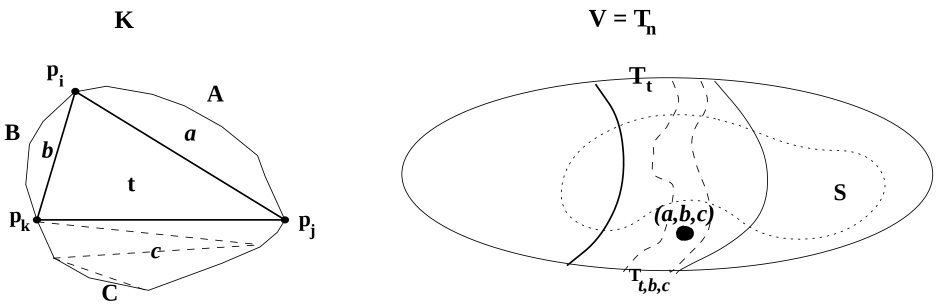

Figure 1: *Triangulations with central triangle t. Subpolygon C is triangulated by c. $\tau = (a,b,c)$ is in $\overline{S} \cap T_t$. $T_{t,b,c}$ denotes triangulations in T_t with b triangulating B and c triangulating C.*

$p_i p_{i+n/2}$ has four neighbors, $p_{i-1}p_{i-1+n/2}$; $p_{i+1}p_{i+1+n/2}$, $\Delta p_{i-1}p_{i+1}p_{i+n/2}$, and $\Delta p_i p_{i-1+n/2} p_{i+1+n/2}$. A central triangle has at most 6 neighbors in $G(C_n)$.

For each $t \in C_n$ we let T_t denote the triangulations τ which have t as their central triangle. The sets T_t partition V. Given a central triangle $t = \Delta p_i p_j p_k$, $i < j < k$, let A denote the subpolygon $p_i, p_{i+1}, \ldots, p_j, p_i$ defined by $p_i p_j$, B, the subpolygon defined by $p_i p_k$, and C, the subpolygon defined by $p_j p_i$ (see Figure 1). We write T_A, T_B, and T_C for the triangulations of A, B, and C, respectively. Observe that there are two degenerate cases here. The first is when n is even and t is degenerate. In this case we think of A as a polygon with one vertex and $|T_A| = 1$. In the other, n is odd and t contains an edge of K. In this case we think of A as a polygon with two vertices and again $|T_A| = 1$. We note that each triple $a \in T_A$, $b \in T_B$, $c \in T_C$ gives a distinct triangulation $\tau = (a,b,c) \in T_t$, and that

$$T_t = \bigcup_a \bigcup_b \bigcup_c (a,b,c),$$

so $|T_t| = |T_A||T_B||T_C|$. In fact suppose that $\|p_i p_j\| = q$, $\|p_j p_k\| = r$, and that $\|p_k p_i\| = s$, where $p + q + r = n$ and $n/3 \leq \max(q,r,s) \leq n/2$. This means that A has $q+1$ sides so (1) gives $|T_A| = \gamma_{q-1}$. Similarly $|T_B| = \gamma_{r-1}$ and $|T_C| = \gamma_{s-1}$, and we have

$$|T_t| = \gamma_{q-1}\gamma_{r-1}\gamma_{s-1}.$$

This expression is maximized when $q = 0$ and $r = s = n/2$ (a degenerate central triangle) and minimized when $q = r = s = n/3$. Since

$$\gamma_j \sim \frac{4^j}{\sqrt{\pi j^3}}$$

this implies

$$\frac{9}{n^3} \leq \frac{|T_t|}{|V|} \leq \frac{5}{n^{3/2}}. \tag{2}$$

One consequence is that for $n \geq 20$, $|T_t| \leq |V|/16$ for degenerate central triangles.

A diagonal $p_i p_{i+2}$ of length 2 defines an *ear at* p_{i+1}, and every $\tau \in V$ has at least 2 (and at most $n/2$) ears. Define

$$E_i \equiv \{\tau \in V : \tau \text{ has an ear at } p_i\},\ i = 1, \ldots, n,$$

the triangulations with ears at vertex p_i. Clearly $|E_i| = |T_{n-1}|$ so from (1),

$$|E_i| = \frac{1}{4}|T_n|\left(1 + \frac{3}{2n-5}\right). \tag{3}$$

The edge $p_0 p_1$ of K must be in some triangle of the triangulation, say $\Delta p_0 p_1 p_i$. According to (3), for about 1/4 of the triangulations in T_n, $p_i = 2$ (this puts an ear at p_1), and in about 1/4 of the triangulations $p_i = n-1$ (an ear at p_0). If a triangulation τ contains triangle $t = \Delta p_0 p_1 p_i$ we write $APEX_n(\tau) = \min(i, n-i)$, $2 \leq i \leq n-1$, the "distance" of the apex of t from its base, $p_0 p_1$. By (1), the number of triangulations with $APEX = i$ is

$$\gamma_{i-2}\gamma_{n-i-1}.$$

This implies that the average

$$A = \frac{1}{|T_n|}\sum_{\tau \in T_n} APEX_n(\tau) = O(\sqrt{n}); \tag{4}$$

also the variance $V = \sum_{\tau \in T_n}(A - APEX_n(\tau))^2 = O(n^{3/2})$. By symmetry, in a random triangulation, the apex of the triangle containing polygon edge $p_j p_{j+1}$ is expected to have distance $O(\sqrt{n})$ from that edge.

Finally, given central triangles t and t',

1. There is a path $P : t = u_0, u_1, \ldots, u_k = t'$ in $G(C_n)$ with no interior u_i degenerate, and

2. $|T_{u_i}|$ is minimized on one of the endpoints unless both t and t' have a side of length ≤ 2. In this case we can guarantee that $\min(|T_{u_i}|, u_i \in P) \geq |T_t|/64$.

3 A Lower Bound on the Mixing Rate

We prove that the mixing rate is at least $\Omega(n^{3/2})$. Let $t = \Delta p_1 p_{n/2} p_{3n/4}$ be the initial central triangle and we start the chain at a random element $\tau \in T_t$. The walk continues while the central triangle avoids the interval I from $p_{2n/10}$ to $p_{3n/10}$, an interval of length $n/10$ (see Fig. 2). The important fact is that the subset $S \subset V$ whose central triangles have at least one vertex in I satisfies

$$P = \frac{|S|}{|V|} \geq c,$$

for some constant $c > 0$. Let N denote the number of steps of $MC(n)$ - starting at τ - before the walk hits a triangulation in S. We will show that

Lemma 1 *$E(N) \geq \Omega(n^{3/2})$.*

From this it is easy to show that for any function $w(n) \uparrow \infty$,

$$\text{Prob}\left\{N < \frac{n^{3/2}}{w(n)}\right\} \to 0.$$

Since S has probability $P > 0$, the walk cannot be within ε of the uniform distribution in $n^{3/2}/w(n)$ steps. With a little more effort we can establish stronger concentration for N and eliminate $w(n)$.

Proof: The initial central triangle t is on the left half of K. The edge $\beta = p_1 p_{n/2}$ is called the *bounding* edge: no vertex of t is to the right of it. Furthermore, the part of the triangulation to the right of the bounding edge is *random.* β is flipped with probability $1/(n-3)$ so $\geq n$ steps of the chain are expected before this flip. At the moment β is flipped, it is in a triangle that is part of a random triangulation of the right side of K, call it $\Delta p_1 p_i p_{n/2}$. If $i \in I$ the walk stops. Otherwise one of the subpolygons $p_1, \cdots, p_i, p_1$ or $p_i, \cdots, p_{n/2}, p_i$ includes I and the corresponding diagonal ($p_1 p_i$ or $p_i p_{n/2}$) becomes the new bounding edge β_1. According to (4), we expect one of the endpoints of the bounding edge to move towards I by at most $d\sqrt{n}$ vertices.

We apply the same argument inductively to the current bounding edge β_k. It will always be the case that (see Fig. 2)

1. The part of the current triangulation to the right of β_k is random.

2. The central triangle of the current triangulation is to the left of β_k.

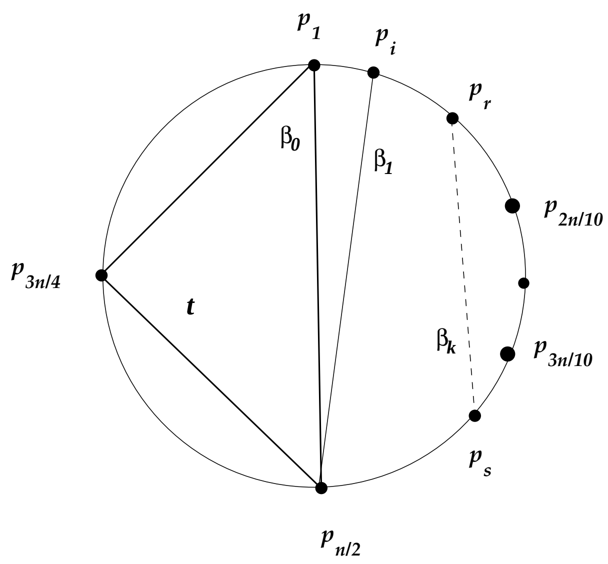

Figure 2: The Initial Central Triangle. $\beta_0 = p_1 p_{n/2}$ is the initial bounding edge; β_k is the edge after it has been flipped k times.

3. At least n steps of the chain are expected before β_k is flipped.
4. If the walk continues, the new bounding edge β_{k+1} has one if its vertices closer to I than the corresponding vertex of β_k, and by an amount $\leq d\sqrt{n}$ vertices, expected.

Suppose the walk stops at the M^{th} step of the chain. The previous facts imply that $E(M) \geq d^{-1}n^{3/2}$. Since no central triangle has yet had a vertex in I, $N \geq M$ and we are finished. ∎

4 The Conductance Bound

The conductance $\Phi(G_n)$ is the minimum over all sets $S \subset V$, $|S| \leq |\overline{S}|$, of

$$\frac{1}{n-3}\frac{|\text{edges } S \to \overline{S}|}{|S|}.$$

Given $S \subset T_n$, write $r = |S|/|T_n|$ as its fraction of $V = T_n$. Theorem 1 asserts that for any $S \subset T_n$ there is a matching from S to $\overline{S}$ with

$$\geq \frac{|S||\overline{S}|}{|T_n|n^{\beta}} \tag{5}$$

edges of G_n and this implies that the conductance is greater than $1/(2n^{\beta+1})$.

For a central triangle $t \in C_n$ write $S_t = S \cap T_t$ and $r_t = |S_t|/|T_t|$ as its fraction of T_t. To prove (5) we need the following facts.

Lemma 2 *For any central triangle t with $r_t \leq 12/23$, there is a matching with at least $\frac{2|S_t|}{n^\beta}$ edges from S_t to $\overline{S} \cap T_t$.*

Lemma 3 *Consider central triangles t_1 with fraction $r_1 = |S_{t_1}|/|T_{t_1}|$ and t_2 with $r_2 = |S_{t_2}|/|T_{t_2}|$, and a path P (respectively cycle C) containing them in $G(C_n)$. Let MIN be the minimum of $|T_t|$ over all t on P (resp C). Then if no triangle internal to P (resp. on C) is degenerate, there is a matching between $S \cap (\cup_{t \in P(resp\ C)} T_t)$ and $\overline{S} \cap (\cup_{t \in P(resp\ C)} T_t)$ with*

$$\geq \frac{31|r_1 - r_2|MIN}{n^8} \left(resp\ \frac{40|r_1 - r_2|MIN}{n^8} \right)$$

edges. If there is a degenerate triangle interior to P (resp. on C) then we can find a matching with at least

$$\geq \frac{7|r_1 - r_2|MIN}{n^8} \left(resp\ \frac{8|r_1 - r_2|MIN}{n^8} \right)$$

edges.

Lemma 4 *Consider adjacent triangles t_1 and t_2 in $G(C_n)$. Write $r_i = |S \cap T_{t_i}|/|T_{t_i}|$. If one of t_1 or t_2 is not degenerate then there is a matching between $S \cap (T_{t_1} \cup T_{t_2})$ and $\overline{S} \cap (T_{t_1} \cup T_{t_2})$ with*

$$\frac{63|r_1 - r_2| \min(|T_{t_1}|, |T_{t_2}|)}{n^8}$$

edges. If both t_1 and t_2 are degenerate then there is such a matching with at least

$$\frac{15|r_1 - r_2| \min(|T_{t_1}|, |T_{t_2}|)}{n^8}$$

edges.

Proof of Theorem 1: We prove (5) with $\beta = 11$. Because $r = |S|/|T_n| \leq 1/2$, the asserted size of the matching is less than 1 unless $\gamma_{n-2} > 2n^{11}$, so we proceed by induction on n, now assuming that $n \geq 35$.

If there were big matchings from $S \cap T_t$ to $\overline{S} \cap T_t$ for enough of the T_t we would be done. Let $Big = \{t \in C_n : r_t > 12/23\}$ and $T_{Big} = \bigcup_{t \in Big} T_t$. Applying Lemma 2 (the proof is based on a straightforward induction argument (which is deferred to an appendix) to central triangles *not* in Big and summing, we find there is a matching of size

$$2 \sum_{t \notin Big} \frac{|S \cap T_t|}{n^\beta} = 2\frac{|S \cap \overline{T_{Big}}|}{n^\beta}.$$

The right hand side exceeds the asserted bound unless $|S \cap \overline{T_{Big}}| < |S|/2$, so we may assume $|S \cap T_{Big}| \geq |S|/2$.

This implies that there is a central triangle t_1 for which $r_{t_1} \geq 12/23$. Also, since $r \leq 1/2$, there is a $t_2 \in C_n$ with $r_{t_2} \leq 1/2$ and consequently we have a pair of central triangles with $|r_{t_1} - r_{t_2}| \geq 1/46$. We apply the second bound from Lemma 3. By (2), $MIN \geq 9|V|/(n^3)$, so there is a matching with at least

$$\frac{63|r_{t_1} - r_{t_2}||V|}{n^{11}}$$

edges. This says there is a matching with more than $63|S|/(23n^{11})$ edges.$\blacksquare$

Proof of Lemma 3: Take a 2 colouring of the edges of the path (respectively three colouring of the cycle). Choose the colour class which maximizes $\sum_{t_a t_b \in colour\ class} |r_a - r_b|$. To get the desired result, sum over the edges of this colour class, applying Lemma 4 to each edge. $\blacksquare$

Remark 1: Write $t_1 = \Delta p_i p_j p_k$ and $t_2 = p_i p_j p_{k'}$, $|k - k'| = 1$. If there were a big matching linking T_{t_1} to T_{t_2} then Lemma 4 would be true even without the $\frac{63}{n^8}$ factor. Unfortunately this never occurs; T_{t_1} and T_{t_2} are rarely even the same size. However there is a matching linking the subset Y_1 of T_{t_1} in which $p_{k'}$ is an ear in the quotient polygon containing p_k with the subset Y_2 of T_{t_2} in which p_k is an ear in the quotient polygon containing $p_{k'}$. In fact the edges between these two sets form a matching. Furthermore, Y_1 contains about a quarter of T_{t_1} and Y_2 contains about a quarter of T_{t_2}. So, if $\frac{|S \cap Y_i|}{|Y_i|}$ were equal to r_i then Lemma 4 would be true with $63/n^8$, replaced by $\frac{1}{4}$. The only dificulty in the proof occurs when either r_1 is significantly different from $\frac{|S \cap Y_1|}{|Y_1|}$ or r_2 is significantly different from $\frac{|S \cap Y_2|}{|Y_2|}$.

To handle this situation, we apply the following lemma, which may have independent interest, to the two appropriate quotient polygons:

Fix m and write $S(m) = S \cap E_m$ for the elements of S with ears at p_m, To estimate the number of edges entering $S(m)$, define the dispersion

$$D_m(S) = \left| |S(m)| - |S| \frac{|E_m|}{|T_n|} \right|. \tag{6}$$

This measures the deviation from the expected number of elements of S with ears at p_m, assuming S were a randomly chosen set of size $|S|$.

Lemma 5 (The "Ear Lemma") *For any $m = 1, \ldots, n$ there is a matching from S to $\overline{S}$ with at least $D_m(S)/n^8$ edges of G.*

Proof: We recursively prove Lemmas 4 and 5 in the following way: We prove Lemma 4 for any n assuming Lemmas 4 and 5 for all $n' < n$. We prove Lemma 5 for any n assuming both lemmas for $n' < n$ and Lemma 4 for n. The proofs proceed by induction on central triangles. Here is a sketch.

We can take $|S(m)| \geq |S||E_m|/|T_n|$ in (6) since $D_m(S) = D_m(\overline{S})$ and the absolute value is redundant in one of these two cases. Given $t = \Delta p_i p_j p_k \in C_n$, the *ring* $R = R(t)$ is the set

$$\{t, \Delta p_{i+1} p_{j+1} p_{k+1}, \ldots, \Delta p_{i-1} p_{j-1} p_{k-1}\}$$

of rotations of t (subscripts always $\bmod(n)$). Write

$$T_R = \bigcup_{t \in R} T_t$$

and $S_R = S \cap T_R$, and let $\mathcal{R}$ be the set of all rings. A ring is *non-degenerate* if it consists of non-degenerate triangles.

For each $\ell = 1, \ldots, \lfloor n/3 + 1 \rfloor$ the *level* X_ℓ is the set of $t \in C_n$ such that the minimum length quotient polygon has size ℓ, and we write

$$T_{X_\ell} = \bigcup_{t \in X_\ell} T_t$$

and $S_{X_\ell} = S \cap T_{X_\ell}$. We note that X_1 is empty if ℓ is odd and X_2 is empty if ℓ is even. We need to treat X_1 and X_2 differently from X_ℓ, $\ell \geq 3$, and we call then degenerate levels. Finally write for each $t \in C_n$, ring R, and ℓ

$$r = \frac{|S|}{|T_n|}, \quad r_t = \frac{|S_t|}{|T_t|}, \quad r_R = \frac{|S_R|}{|T_R|}, \quad r_\ell = \frac{|S_{X_\ell}|}{|T_{X_\ell}|},$$

and observe that

$$D_m(S) = \sum_{t \in C_n} \Big(|S_t \cap S(m)| - \frac{|S|}{|T_n|} |T_t \cap E_m| \Big). \tag{7}$$

If p_m is a vertex of $t = \Delta p_i p_j p_k$ then $|T_t \cap E_m| = 0$. Otherwise, letting λ_t denote the size of the quotient subpolygon containing p_m and using (3), it is $|T_t|(1 + 3/(2\lambda_t - 5))/4$, and we have

$$D_m(S) = \sum_{t \in C_n} \Big(|S_t \cap S(m)| - \frac{r|T_t|}{4} (1 + \frac{3}{2\lambda_t - 5}) \Big),$$

where we understand that $\lambda_t = 1$ if t contains p_m. Clearly

$$D_m(S) = \sum_{t \in C_n} \Big(|S_t \cap S(m)| - \frac{r_t|T_t|}{4} (1 + \frac{3}{2\lambda_t - 5}) \Big)$$

$$+ \sum_{\ell=1}^{n/3} \sum_{R \in X_\ell} \sum_{t \in R} [(r_t - r_R) + (r_R - r_\ell) + (r_\ell - r)] \frac{1}{4} |T_t| (1 + \frac{3}{2\lambda_t - 5}),$$

and rewriting the second summations,

$$D_m(S) = \sum_{t \in C_n} \Big(|S_t \cap S(m)| - \frac{r_t|T_t|}{4} (1 + \frac{3}{2\lambda_t - 5}) \Big)$$

$$+ \frac{1}{4} \sum_{R \in \mathcal{R}} \sum_{t \in R} (r_t - r_R) |T_t| (1 + \frac{3}{2\lambda_t - 5}) \tag{8}$$

$$+ \frac{1}{4} \sum_{\ell=1}^{n/3} \sum_{R \in X_\ell} \sum_{t \in R} (r_R - r_\ell) |T_t| (1 + \frac{3}{2\lambda_t - 5}) + \frac{1}{4} \sum_{\ell=1}^{n/3} \sum_{R \in X_\ell} \sum_{t \in R} (r_\ell - r) |T_t| (1 + \frac{3}{2\lambda_t - 5}).$$

We deal with these four summation terms one at a time. The idea is that if any are large, then the lemma is already true. On the other hand they cannot all be small.

Let σ_1 be the value of the first summation in (8) and we write

$$\sigma_1 = \sum_{t \in C_n} D^t(S),$$

where

$$D^t(S) = |S_t \cap S(m)| - \frac{r_t|T_t|}{4} (1 + \frac{3}{2\lambda_t - 5}).$$

We can interpret $D^t(S)$ as the dispersion in the quotient subpolygon containing p_m. Using Lemma 5 inductively and noting that the subpolygon has size $\lambda_t + 1 \leq n/2$, $D^t(S)$ accounts for at least $D^t(S)/(n/2)^8$ edges from S_t to $\overline{S} \cap T_t$. Summing over C_n reveals that $D_m(S)$ accounts for at least $2^8 \sigma_1 / n^8$ edges from S to $\overline{S}$, so unless $\sigma_1 \leq D_m(S)/2^8$, the ear lemma is already true.

Next consider

$$\sigma_2 = \frac{1}{4} \sum_{R \in \mathcal{R}} \sum_{t \in R} (r_t - r_R) |T_t| (1 + \frac{3}{2\lambda_t - 5}),$$

the second summation term in (8), and consider a non-degenerate ring R. It consists of rotations of a triangle t with sides of length $a \leq b \leq c$, $a+b+c = n$, $n/3 \leq c \leq n/2$. For each t in R either (i) p_m is in t or p_m is in the quotient polygon of size (ii) $a+1$, (iii) $b+1$, or (iv) $c+1$. For each ring R, the term in σ_2 is

$$\sum_{t \in R} (r_t - r_R) \frac{|T_t|}{4} (1 + \frac{3}{2\lambda_t - 5}) = \sum_{t \in R} r_t \frac{|T_t|}{4} - r_R \sum_{t \in R} \frac{|T_t|}{4} + \sum_{t \in R} \frac{3}{2\lambda_t - 5} \frac{|T_t|}{4} (r_t - r_R)$$

$$= \frac{|S_R|}{4} - \frac{|S_R|}{4} + \sum_{t \in R} \frac{3}{2\lambda_t - 5} \frac{|T_t|}{4} (r_t - r_R).$$

Now, for each non-degenerate ring R, we let

$$D_R(S) = \frac{1}{4} \sum_{t \in R} \frac{3}{2\lambda_t - 5} |T_t| (r_t - r_R)$$

be the term in the σ_2 sum which corresponds to R. For each non-degenerate R, we can find a cycle C_R in $G(C_n)$ containing no degenerate triangles, which contains all of R, and has exactly $3|R|$ vertices all of which are either in R or adjacent to an element of R. We call C_R, the *extended ring* for R. Each central triangle is in at most 7 extended rings since it has at most six neighbours in $G(C_n)$. Let $T_{C_R} = \bigcup_{t \in C_R} T_t$.

Claim 1 *(i) For each non-degenerate ring R, there is a matching between $S \cap T_{C_R}$ and $\overline{S} \cap T_{C_R}$ with at least $\frac{21}{20}$ $D_R(S)$ edges. (ii) For the (at most one) degenerate ring R that there is a matching from $S \cap T_R$ to $\overline{S} \cap T_R$ with at least $\sum_{t \in R} \frac{3}{2\lambda_t - 5} |T_t| (r_t - r_R)$ edges.*

This claim is proved by choosing t_1 and t_2 which respectively maximize and minimize r_t over C_R and then applying Lemma 3 to t_1,t_2, and C_R. We omit the details. Since each central triangle is in at most seven extended rings, the union of these matchings over the non-degenerate triangles has maximum degree seven, and hence has an eight colouring. Adding the disjoint matching from the degenerate ring to the largest color class, and summing the contributions of all the rings, we see that if σ_2 accounts for more than $1/2$ of $D_m(S)$, Lemma 5 is already true.

Next we can apply Lemma 3 in a similar but more complicated fashion to show that the third term in (8), σ_3, must be less than $D_m(S)/400$, and finally, that σ_4, the fourth summation in (8), is less than $5D_m(S)/12$. Putting these statements together, the ear lemma is already true unless

$$D_m(S) \leq \frac{D_m(S)}{2^8} + \frac{D_m(S)}{2} + \frac{D_m(S)}{400} + \frac{5D_m(S)}{12} < D_m(S).$$

an impossibility that completes the proof. ∎

Proof of Lemma 4: We apply the Ear Lemma inductively,assuming it is true for polygons with at most $n-1$ sides. Suppose first the two triangles are non-degenerate. Let t_1, t_2, Y_1, Y_2 be as in Remark 1. Let $r_i' = |S \cap Y_i|/|Y_i|$. If $|r_1' - r_2'| \geq 64|r_1 - r_2|/n^8$, then the result holds by considering the matching linking Y_1 and Y_2. Otherwise, $|r_1 - r_1'| + |r_2 - r_2'| \geq (1 - \frac{65}{n^8})|r_1 - r_2|$. Since Y_i is just E_m for some ear in one of the quotient polygons defined by t_i, the result holds by applying the Ear Lemma to these two quotient polygon (we need to sum over all pairs of triangulations of the other polygons but this is straightforward). We get a 2^8 factor bonus because the quotient polygons are smaller. This explains the 63 in Lemma 4 which is not present in the Ear Lemma. If exactly one of the t_i is degenerate, we proceed as above, but now the matching is between a subset of T_{t_1} where we fix an ear in both quotient polygons. However the matching hits all of T_{t_2} so $MIN = |T_{t_2}|$ and the proof technique works. If t_1 and t_2 are both degenerate, then the fact that we consider two ears - not one - accounts for the loss of a factor of 4 in the result. We omit the details. ∎

References

[1] D. Arnold and M. Sleep. Uniform Random Number Generation of n Balanced Parenthesis Strings. *ACM Trans. on Prog. Languages and Systems 2*, 122-128, 1980.

[2] M. Atkinson and J. Sack. Generating Binary Trees at Random. *Inf. Proc. Letters 41*, 21-23, 1992.

[3] L. Devroye, P. Flajolet, F. Hurtado, M. Noy, and W. Steiger. Random Triangulations. manuscript, 1996.

[4] L. Devroye. A Note on the Height of Binary Search Trees. *J.A.C.M. 33*, 489-498, 1986.

[5] L. Devroye. Application of the Theory of Records in the Study of Random Trees. *Acta Informatica 26*, 123-130, 1988.

[6] P. Flajolet and A. Odlyzko. The Average Height of Binary Trees and Other Simple Trees. *J. Comp. Syst. Sciences, 25*, 171-213, 1982.

[7] F. Hurtado and M. Noy. The Graph of Triangulations of a Convex Polygon. *ACM Symposium on Computational Geometry (to appear)*, 1996.

[8] H. Mahmoud. *Evolution of Random Search Trees.* John Wiley, 1992.

[9] A. Meir and J. Moon. On the Altitude of Nodes in Random Trees. *Canad. J. Math. 30*, 997-1015, 1978.

[10] B. Pittel. On Growing Random Binary Trees. *J. Math. Anal. Appl. 103*, 461-480, 1984.

[11] A. Sinclair. Algorithms for Random Generation and Counting: A Markov Chain Approach. *Birkhauser, Boston, 1992*

[12] D. Sleator, R. Tarjan, and W. Thurston. Rotation Distance, Triangulations, and Hyperbolic Geometry. *J. Amer. Math. Soc. 1*, 647-681, 1988.