The adaptable choosability number grows with the choosability number

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Abstract

The adaptable choosability number of a multigraph $G$, denoted $ch_a(G)$, is the smallest integer $k$ such that every edge labeling of $G$ and assignment of lists of size $k$ to the vertices of $G$ permits a list coloring of $G$ in which no edge $e = uv$ has both $u$ and $v$ colored with the label of $e$. We show that $ch_a$ grows with $ch$, i.e. there is a function $f$ tending to infinity such that $\chi(G) \leq f(ch(G))$.

Keywords: adaptable coloring, list coloring

1. Introduction

Hell and Zhu introduced the adaptable chromatic number in [11]. Given a multigraph whose edges are labeled from $[k] = \{1, 2, \ldots, k\}$, the goal is to color the vertices with colors from $[k]$ so that there is no edge $e = uv$ such that $u$ and $v$ are both colored with the label of $e$. A vertex coloring which satisfies this property is called an adaptable vertex coloring. The adaptable chromatic number of a graph $G$, denoted $\chi_a(G)$, is the minimum number $k$ such that every edge labeling of $G$ from $[k]$ permits an adaptable vertex coloring from $[k]$. It has been studied in [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15] (in some cases by a different name).

Note that every proper vertex coloring of a graph $G$ is an adaptable vertex coloring for any edge labeling and thus $\chi_a(G) \leq \chi(G)$. The inequality is tight as there are infinite families of graphs where $\chi_a(G) = \chi(G)$ [10, 11]. These parameters can also be far apart as there are infinite families of graphs where $\chi_a(G) = \Theta \left(\sqrt{\chi(G)}\right)$ (for example, the complete graph [4]). This brings us
to the following question proposed by Hell and Zhu in [11].

**Question.** Is there a function $f$ tending to infinity such that $\chi_a(G) \geq f(\chi(G))$?

As far as we know, the answer may be ‘yes’ with $f(k) = \Theta(\sqrt{k})$; i.e. the complete graph may be asymptotically extremal.

In this paper, we study adaptable list coloring, which is defined naturally in [12]: Given a multigraph $G$, the *adaptable choosability number*, denoted $ch_a(G)$, is the minimum number $k$ such that every edge labeling of $G$ and assignment to each vertex $v$ of a list $L(v)$ of size $k$, there is an adaptable coloring of $G$ from these lists. As with $\chi_a$, it is trivial that $ch_a(G) \leq ch(G)$, where $ch(G)$ is the choosability number. We answer the list coloring version of Hell and Zhu’s question.

**Theorem 1.1.** There is a function $h$ tending to infinity such that $ch_a(G) \geq h(ch(G))$.

Our proof obtains $h(k) = \Theta(\log^{1/5}k)$, but we made no effort to optimize it. As far as we know, we can have $h(k) = \Theta(\sqrt{k})$. We know, however, that $h(k) = O(\sqrt{k})$ since, like with $\chi_a$, the complete graph has $ch_a(K_n) = \Theta(\sqrt{ch(K_n)})$ [12, 14].

The proof of the theorem uses a probabilistic approach and takes advantage of the Chernoff bound [3]. Instead of using the original statement we use the (weaker) version found in [13].

**Chernoff Bound.** For any $0 \leq t \leq np$,

$$\Pr(|\text{BIN}(n,p) - np| > t) < 2e^{-\frac{t^2}{3np}},$$

where $\text{BIN}(n,p)$ is the sum of $n$ independent variables, each equal to 1 with probability $p$ and 0 otherwise.

2. **Proof of Main Theorem**

The proof of Theorem 1.1 closely follows the approach taken by Alon in [1] for a similar result on normal list coloring.

We start by proving the following theorem, where $\delta(G)$ is the minimum degree of $G$. 
Theorem 2.1. There is a function \( g \) tending to infinity such that if \( H \) is a bipartite graph satisfying \( \delta(H) \geq d \) then \( ch_a(H) \geq g(d) \).

Theorem 1.1 easily follows from Theorem 2.1.

Proof of Theorem 1.1. We use the following two well known and easily proved facts:

(i) If \( \delta(G) \geq d \), \( G \) has a bipartite subgraph with minimum degree at least \( \frac{d}{2} \). This can be seen by taking a spanning bipartite subgraph of \( G \) with the maximum number of edges.

(ii) If \( ch(G) \geq k \), then it has a subgraph of minimum degree at least \( k - 1 \). This can be seen by taking a graph where every subgraph has minimum degree at most \( k - 2 \) and iteratively coloring the vertex with minimum degree and then removing it from the graph.

Therefore, the function \( h(k) = g \left( \frac{k-1}{2} \right) \) satisfies the desired properties. \( \square \)

Note that Fact (ii) holds for the coloring number (the maximum of \( \delta(H) + 1 \) over all subgraphs \( H \) of \( G \)) as well, so Theorem 1.1 can be strengthened to show that \( ch_a \) grows with the coloring number.

To prove Theorem 2.1, consider any bipartite graph \( H \) with bipartition \( (A, B) \) where \( |A| \geq |B| \). We will consider lists of size \( s \) taken from a color set of size \( s^5 \). We will show that there is a function \( f(s) \) such that if \( \delta(H) \geq f(s) \), then there is an assignment of lists to vertices and labels to edges such that there is no proper adaptable coloring from these lists. This is sufficient to show that \( ch_a(H) > s \). This clearly is sufficient to prove Theorem 2.1, as we can let \( g = f^{-1} \).

We start with a few helpful definitions. An assignment of lists to \( A \) (resp. \( B \)) is called an \( \text{A-set} \) (resp. \( \text{B-set} \)). Given a \( \text{B-set} \), we say that \( a \in A \) is \textit{supersurrounded} (inspired by “surrounded” from [1]) if every possible list of \( s \) elements from \( [s^5] \) appears in more than \( s^3 \) lists on vertices in \( N(a) \) (the neighborhood of \( a \)). Furthermore, we call the \( \text{B-set} \) \textit{bad} if at least half of the vertices in \( A \) are supersurrounded.

Theorem 2.1 follows directly from the following two lemmas.

Lemma 2.2. If \( \delta(H) \geq d = 36s^5 \binom{s^5}{s} \), then there is a bad \( \text{B-set} \) of lists.
Lemma 2.3. There is an $s_0$ such that for any bad $B$-set $B$, if $s \geq s_0$, there is an assignment of colors to the edges of $H$ and an $A$-set $A$ such that $H$ does not have an acceptable coloring.

Proof of Theorem 2.1. Let $g$ be the inverse of the function $f(s) = 36s^5\binom{s^5}{s}$. We choose a bad $B$-set $B$ according to Lemma 2.2. We choose an $A$-set $A$ and an edge coloring according to Lemma 2.3 such that there is no acceptable coloring from the assigned lists.

Proof of Lemma 2.2. Uniformly at random assign lists to each of the vertices in $B$.

Let $a \in A$ be an arbitrary vertex and let $Y$ be the number of lists which do not appear more than $s^3$ times in $a$’s neighborhood. We will show that the probability that $a$ is not supersurrounded, i.e. that $Y \geq 1$, is less than $1/2$.

To make this computation it will be helpful to consider a single list. Let $S \subseteq [s^5]$ an arbitrary list of size $s$ and let $X$ be the number of neighbors of $a$ whose assigned list is $S$.

Since the lists are assigned uniformly at random, for each neighbor $b$ of $a$, the probability that $b$ is assigned $S$ is $\frac{1}{\binom{s^5}{s}}$. Therefore:

$$\mathbb{E}(X) = \frac{|N(a)|}{\binom{s^5}{s}} \geq \frac{d}{\binom{s^5}{s}} = 36s^5$$

The Chernoff bound yields the following.

$$\Pr \left( X \leq s^3 \right) \leq \Pr \left( X \leq \frac{\mathbb{E}(X)}{2} \right) \leq \Pr \left( |X - \mathbb{E}(X)| > \frac{\mathbb{E}(X)}{2} \right)$$

$$< 2e^{-[\mathbb{E}(X)/2]^2/[3\mathbb{E}(X)]} = 2e^{-\mathbb{E}(X)/12} \leq 2e^{-36s^5/12} = 2e^{-3s^5}$$

Now we can bound the expected value of $Y$ using the linearity of expectation.

$$\mathbb{E}(Y) = \binom{s^5}{s} \Pr (X \leq s^3) < \binom{s^5}{s} 2e^{-3s^5} \leq 2es^5e^{-3s^5} < \frac{1}{2}, \text{ for every } s \geq 1.$$
Now let $Z$ be the number of vertices in $A$ which are supersurrounded. By the linearity of expectation, $E(Z) > \frac{1}{2}|A|$. Thus the probability that $Z \geq \frac{1}{2}|A|$ is positive, and therefore there is a bad $B$-set. □

Proof of Lemma 2.3. Assume that $B$ is a bad $B$-set.

**Step 1:** For each edge $e = ab$ where $a \in A$ and $b \in B$, assign to $e$ a color uniformly at random from $L(b)$.

Consider any $a \in A$ that is supersurrounded. Fix a coloring of $B$ from the lists of $B$.

We will say that a color $c$ is *available* for $a$ if there is no neighbor $b$ of $a$ such that $ab$ is labeled $c$ and $b$ is colored $c$. A coloring of $B$ is extendable to $A$ if every vertex in $A$ has at least one available color in its list. Note that $G$ is colorable if and only if at least one coloring of $B$ is extendable to $A$.

First we note that all but at most $s-1$ colors appear more than $s^2$ times on vertices in the neighborhood of $a$. We can see this by assuming that $c_1, \ldots, c_s$ all appear at most $s^2$ times in $N(a)$. So the list $\{c_1, \ldots, c_s\}$ can only appear in $N(a)$ at most $s \cdot s^2 = s^3$ times. However, as $a$ is supersurrounded, the list appears more than $s^3$ times and thus we have a contradiction.

Let $c$ be a color that appears more than $s^2$ times in $N(a)$. The probability that a color $c$ is available for $a$ is the probability that for every neighbor $b$ of $a$ such that $b$ is colored $c$, the edge $e = ab$ is not labeled $c$. Note that since we are choosing the color for $e$ from $b$’s list of colors, the probability that $e$ is colored the same as $b$ is $1/s$. Therefore:

$$\Pr(c \text{ is available}) < \left(1 - \frac{1}{s}\right)^{s^2} < e^{-s}.$$  

Define $Z$ to be the number of available colors beyond the $s-1$ colors which may appear $s^2$ or fewer times,

$$E(Z) < s^5e^{-s}$$

Using Markov’s Inequality:

$$\Pr(Z \geq 1) \leq E(Z) < s^5e^{-s}.$$  

Now, including the $s-1$ colors which may appear $s^2$ or fewer times, we can with high probability bound the number of available colors as follows.

$$\Pr(\# \text{ available colors for } a \geq s) < s^5e^{-s}. \quad (1)$$
Step 2: For each vertex $a \in A$, uniformly at random choose one of the $\binom{s^5}{s}$ possible lists.

Now, assuming that $a$ is a vertex with fewer than $s$ available colors, we can bound the probability that the list chosen for $a$ has an available color. Since there are at most $s - 1$ colors available for $a$, the probability that a random color $c$ is available to $a$ is at most $\frac{s-1}{s^5}$.

$$\Pr(\text{list chosen for } a \text{ contains an available color}) \leq s \cdot \frac{s-1}{s^5} < \frac{1}{s^3}$$

Therefore, by (1), the probability that $a$ has $s$ or more available colors or the list chosen for $a$ has an available color is less than $s^5 e^{-s} + s^{-3}$. For sufficiently large $s$, this is less than $1/s^2$.

Since $B$ is a bad $B$-set, there are at least $\frac{1}{2}|A|$ supersurrounded vertices. Thus, remembering that $|A| \geq |B|$, we can bound the probability that every supersurrounded vertex has an available color in its list as follows.

$$\Pr\left(\text{every supersurrounded vertex has an available color in its list}\right) < \left(\frac{1}{s^2}\right)^{\frac{1}{2}|A|} = s^{-|A|} \leq s^{-|B|}$$

Let $W$ be the number of colorings of $B$ which are extendable to $A$. Given a $B$-set $B$, there are $s^{|B|}$ possible ways of choosing colors for the vertices in $B$. Thus we can bound the expected value of $W$ as follows.

$$\mathbb{E}(W) < s^{|B|} \cdot s^{-|B|} = 1$$

Since the expected value is less than 1, there must be a choice of an $A$-set and edge colorings such that no coloring of $B$ can be extended to a coloring of $A$.  


