Perfect matchings in random $r-$regular, $s-$uniform hypergraphs.

Colin Cooper[*] Alan Frieze[†] Michael Molloy[‡]
Bruce Reed[§]

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1 Introduction

Let $E = \{X_1, X_2, \ldots, X_m\}$ where the $X_i \subseteq V$ for $1 \leq i \leq m$ are distinct. The hypergraph $G = (V, E)$ is said to be *s-uniform* if $|X_i| = s$ for $1 \leq i \leq m$. Thus, for example, a $2-$uniform hypergraph is a graph. A set of edges $M = \{X_i : i \in I\}$ is a *perfect matching* if

(i) $i \neq j \in I$ implies $X_i \cap X_j = \emptyset$, and

(ii) $\bigcup_{i \in I} X_i = V$.

[*]School of Mathematical Sciences, University of North London. Research done while visiting Carnegie Mellon University

[†]Department of Mathematics, Carnegie-Mellon University. Supported in part by NSF grants CCR9024935 and CCR9225008.

[‡]Department of Mathematics, Carnegie-Mellon University. Supported in part by NSF grant CCR9225008.

[§]Department of Mathematics, Carnegie-Mellon University

One of the most interesting and difficult problems in probabilistic combinatorics can be described as follows: suppose that the X_i are chosen independently at random from the $\binom{|V|}{s}$, $s-$subsets of V. For what value of m, the number of edges, is it likely that G will contain a perfect matching? When $s=2$, this was solved by Erdös and Rényi [4]. For $s \geq 3$ we only have the fairly loose results of Schmidt and Shamir [9].

Putting $|V| = sn$ it is reasonable to make the following:

CONJECTURE. Assume s is a positive integer constant and $m = n(\log n + \log s + c_n)$ then

$$\lim_{n\to\infty} \mathbf{Pr}(G \text{ has a perfect matching }) = \begin{cases} 0 & c_n \to -\infty, \\ e^{-e^{-c}} & c_n \to c, \\ 1 & c_n \to \infty. \end{cases}$$

The right-hand side of the above expression is simply the limiting probability that $\bigcup_{i=1}^m X_i = V$. The case $s=2$ was dealt with in [4].

A related and special case of the problem is that of packing vertex disjoint copies of a fixed graph H in a random graph G. The existence of perfect packings was solved completely by Łuczak and Ruciński [5] for the case when H is a tree. Less precise results were obtained by Ruciński [8] for arbitrary graphs.

For $v \in V$, let $d_H(v) = |\{i : v \in X_i\}|$ be the *degree* of v. H is $r-$*regular* if $d_H(v) = r$ for all $v \in V$. Let now $V = [sn]$, where $[k] = \{1, 2, \ldots, k\}$ for all positive integers k. Let $\mathcal{G} = \mathcal{G}(n, r, s) = \{G = (V, E) : G$ is $r-$regular and $s-$uniform $\}$. Let $G = G_{n,r,s}$ be chosen uniformly at random from $\mathcal{G}$. In this paper we prove

Theorem 1 *Suppose r, s are fixed positive integers, then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,r,s} \textit{ has a perfect matching }) = \begin{cases} 0 & s > \sigma_r \\ 1 & s < \sigma_r \end{cases}$$

where

$$\sigma_r = \frac{\log r}{(r-1)\log\left(\frac{r}{r-1}\right)} + 1.$$

□

[Note that σ_r is always non-integral and so this result is best possible.]

Next let $f(s) = \min\{r : s < \sigma_r\}$. Thus $f(s)$ gives the threshold in terms of degree for a s-uniform hypergraph to almost surely have a perfect matching. We have computed the first few values of $f(s)$ and they are given in Table 1. For s large $f(s)$ is approximately e^{s-1}.

s	2	3	4	5	6	7	8	9	10
$f(s)$	3	7	19	52	146	401	1094	2979	8126

Table 1:

To prove the theorem, we make use of a remarkable new approach due to Robinson and Wormald [6] and [7]. Although new to probabilistic combinatorics, we will see that their method is in fact an *Analysis of Variance* technique with a clever partition of the probability space based on the number of small cycles.

Since the case $s = 2$ is well known, we will assume that $s \geq 3$ from now on.

To prove our theorem, we need a suitable probabilistic model for generating

$\mathcal{G}(n,r,s)$. We will use a natural generalisation of the Configuration Models of Bender and Canfield [2] or Bollobás [3], which we now describe.

2 Configurations

Let $W_v = \{v\} \times [r]$ for $v \in V = [sn]$ and $W = \bigcup_{v \in V} W_v$. Each W_v should be regarded as a *block* of r *fractional edges* for each $v \in V$, thus generalising the concept of half-edges arising from the use of configurations in the context of graphs. In this context, a *configuration* is a partition of W into $m = rn$ subsets of size s. Let $\Omega = \Omega(n,r,s)$ be the set of all such configurations, and let $F = F(n,r,s)$ be chosen randomly from Ω.

For $x = (v,i) \in W$ we let $V(x) = v$. If $F \in \Omega$ and $S \in F$ we let $V(S) = \{V(x) : x \in S\}$. We define the multigraph $\gamma(F) = (V, \{V(S) : S \in F\})$.

F is *simple* if $S \in F$ implies $|V(S)| = s$ and distinct $S_1, S_2 \in F$ have $V(S_1) \neq V(S_2)$. Thus $\gamma(F)$ is s-uniform if and only if F is simple. A routine calculation shows that

$$\lim_{n\to\infty} \mathbf{Pr}(\exists S_1, S_2 \in F \text{ with } V(S_1) = V(S_2)) = 0.$$

The main properties we need are

(A) each $G \in \mathcal{G}$ arises from precisely $(r!)^{sn}$ configurations F.

(B) $\lim_{n\to\infty} \mathbf{Pr}(F \text{ is simple }) = e^{-(s-1)(r-1)/2}$ (see Lemma 2 below).

A perfect matching of F is then a set $\{S_i : i \in I\} \subseteq F$ such that

(i) $|V(S_i)| = s, i \in I$,

(ii) $i, j \in I, i \neq j$ implies $V(S_i) \cap V(S_j) = \emptyset$, and

(iii) $\bigcup_{i \in I} V(S_i) = V$.

Thus $F \in \Omega^0$ has a perfect matching if and only if $\gamma(F)$ has a perfect matching. Furthermore, Theorem 1 will follow immediately from (**A**) and (**B**) above and

Theorem 2

$$\lim_{n\to\infty} \mathbf{Pr}(F \textit{ has a perfect matching }) = \begin{cases} 0 & s > \sigma_r \\ 1 & s < \sigma_r \end{cases}$$

□

3 Outline Proof of Theorem 2

We use the notation $\alpha \approx \beta$ to mean $\alpha = (1 + o(1))\beta$ where the $o(1)$ term tends to zero as n tends to infinity. All subsequent inequalities are only claimed to hold for sufficiently large n.

Suppose that F is chosen randomly from Ω. Let $Z(F)$ denote the number of perfect matchings in F. We will prove the following lemma in Section 4.

Lemma 1

$$\mathbf{E}(Z) \approx \sqrt{s}\left(r\left(\frac{r-1}{r}\right)^{(s-1)(r-1)}\right)^n, \tag{1}$$

$$\frac{\mathbf{E}(Z^2)}{\mathbf{E}(Z)^2} \approx \sqrt{\frac{r-1}{r-s}}, \qquad \textit{if } s > r. \tag{2}$$

Notice that the first (easy) part of Theorem 1 now follows immediately since the righthand side of (1) tends to zero exponentially fast when $s > \sigma_r$.

To apply the Analysis of Variance technique, we have to decide on a partition of Ω. We proceed analogously to Robinson and Wormald. For the moment let b, x be arbitrarily large *fixed* positive integers. We now define a $k-$cycle of F for integer $k \geq 1$.

$k = 1$: $S \in F$ is a $1-$cycle if $|V(S)| < s$.

$k = 2$: $S_1, S_2 \in F$ form a $2-$cycle if $|V(S_1) \cap V(S_2)| \geq 2$.

$k \geq 3$: $S_1, S_2, \ldots, S_k \in F$ form a $k-$cycle if there exist $v_1, v_2, \ldots, v_k \in V$ such that $v_i \in V(S_i) \cap V(S_{i+1})$ for $1 \leq i \leq k$, $(S_{k+1} \equiv S_1)$.

Observe that F is simple if and only if it has no 1-cycles and yields no repeated edges.

Next let C_k denote the number of $k-$cycles of F for $k \geq 1$. For $\mathbf{c} = (c_1, c_2, \ldots, c_b) \in N^b$, where $N = \{0, 1, 2, \ldots\}$, let $\Omega_{\mathbf{c}} = \{F \in \Omega : C_k = c_k, 1 \leq k \leq b\}$. Let

$$\lambda_k = \frac{((s-1)(r-1))^k}{2k}.$$

Lemma 2 *Let* $\mathbf{c}$ *be fixed, then*

$$\pi_c = \mathbf{Pr}(F \in \Omega_{\mathbf{c}}) \approx \prod_{k=1}^{b} \frac{\lambda_c^{c_k} e^{-\lambda_k}}{k!}.$$

Now define

$$S(x) = \{\mathbf{c} \in N^b : |c_k - \lambda_k| \leq x\lambda_k^{2/3}, 1 \leq k \leq b\},$$

and

$$\overline{\Omega} = \bigcup_{\mathbf{c} \notin S(x)} \Omega_{\mathbf{c}}.$$

Let

$$\overline{\pi} = \mathbf{Pr}(F \in \overline{\Omega}).$$

For $\mathbf{c} \in N^b$ let

$$E_{\mathbf{c}} = \mathbf{E}(Z \mid F \in \Omega_{\mathbf{c}})$$

and

$$V_{\mathbf{c}} = \mathbf{Var}(Z \mid F \in \Omega_{\mathbf{c}}).$$

Then we have

$$\mathbf{Var}(Z) = \sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}} + \sum_{\mathbf{c} \in N^b} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 - \mathbf{E}(Z)^2. \tag{3}$$

The following two lemmas contain the most important observations. Lemma 3 shows that most of the variance can be explained by the *variance between groups* and Lemma 4 shows that for most groups, the group mean is large.

Lemma 3 *If x is sufficiently large then*

$$\sum_{\mathbf{c} \in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 \geq \left(1 - be^{-3\gamma x}\right)\left(1 - \left(\frac{s-1}{r-1}\right)^b\right)\left(\sqrt{\frac{r-1}{r-s}}\right)\mathbf{E}(Z)^2.$$

where γ is as in Lemma 4 □

Lemma 4 *For all sufficiently large x*

(a) $\overline{\pi} \leq e^{-\alpha x}$ for some absolute constant $\alpha > 0$.

(b) $\mathbf{c} \in S(x)$ implies

$$E_{\mathbf{c}} \geq e^{-(\beta + \gamma x)} \mathbf{E}(Z),$$

for some absolute constants $\beta, \gamma > 0$. □

Hence we have from (3),

$$\sum_{\mathbf{c}\in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}} \leq \delta \mathbf{E}(Z)^2, \tag{4}$$

where $\delta = \left(be^{-3\gamma x} + (\frac{s-1}{r-1})^b\right)\sqrt{\frac{r-1}{r-s}}$. The rest is an application of the Chebycheff inequality. Define the random variable $\hat{Z}(F)$ by

$$\hat{Z}(F) = E_{\mathbf{c}}, \text{ if } F \in \Omega_{\mathbf{c}}.$$

Then for any $t > 0$

$$\begin{aligned} \mathbf{Pr}(|Z - \hat{Z}| \geq t) &\leq \mathbf{E}((Z-\hat{Z})^2/t^2) \\ &= \sum_{\mathbf{c}\in N^b} \pi_{\mathbf{c}} V_{\mathbf{c}}/t^2 \\ &\leq \delta \mathbf{E}(Z)^2/t^2, \end{aligned} \tag{5}$$

where the last inequality follows from (4).

Now put $t = e^{-(\beta+\gamma x)}\mathbf{E}(Z)/2$ where β, γ are from Lemma 4. Applying Lemma 4 we obtain

$$\begin{aligned} \mathbf{Pr}(Z \neq 0) &\geq \mathbf{Pr}(Z \geq e^{-(\beta+\gamma x)}\mathbf{E}(Z)/2) \\ &\geq \mathbf{Pr}(|Z-\hat{Z}| \leq t \wedge (F \notin \overline{\Omega})) \\ &\geq 1 - 4\delta e^{2(\beta+\gamma x)} - \overline{\pi} \end{aligned}$$

Hence, using Lemma 4

$$\lim_{n\to\infty} \mathbf{Pr}(Z=0) \leq \left(4be^{2\beta-\gamma x} + 4\left(\frac{s-1}{r-1}\right)^b e^{2(\beta+\gamma x)}\right)\sqrt{\frac{r-1}{r-s}}. \tag{6}$$

This is true for all b, x and so $\lim_{n\to\infty}\mathbf{Pr}(Z=0)$ must in fact be zero, proving Theorem 2, (putting $b = x^2$ and x arbitrarily large makes the right-hand side of (6) arbitrarily small).

4 Moments

First of all let

$$\psi_s(m) = \frac{(sm)!}{m!(s!)^m}$$

denote the number of ways of partitioning $[sm]$ into m s-sets. Then for any $k \geq 0$,

$$\begin{aligned}
\mathbf{Pr}(F \text{ contains } k \text{ given triples}) &= \frac{\psi_s(rn-k)}{\psi_s(rn)} \\
&\approx \frac{(s!)^k (rn)^k}{(srn)^{sk}}, \qquad \text{if } k \text{ is fixed}
\end{aligned}$$

We can then compute

$$\begin{aligned}
\mathbf{E}(Z) &= \psi_s(n) r^{sn} \frac{\psi_s((r-1)n)}{\psi_s(rn)} \\
&\approx \sqrt{s}\left(r\left(\frac{r-1}{r}\right)^{(s-1)(r-1)}\right)^n.
\end{aligned}$$

on using Stirling's Formula. Here $\psi_s(n)r^{sn}$ counts the number of distinct possible perfect matchings.

We can assume from now on that $s < \sigma_r$. Next we have

$$\mathbf{E}(Z^2) = \mathbf{E}(Z) \sum_{k=0}^{n} \binom{n}{k} \psi_s(n-k)(r-1)^{s(n-k)} \psi_s(rn-2n+k)/\psi_s((r-1)n). \quad (7)$$

Explanation: we choose a fixed perfect matching M_0 and compute the probability that F contains a perfect matching M given it contains M_0. Summing over M_0 accounts for $\mathbf{E}(Z)$. The parameter k denotes the number of s-tuples common to M and M_0. $\binom{n}{k}$ counts the number of ways of choosing these. There are $\psi_s(n-k)(r-1)^{s(n-k)}$ possible completions. The remaining terms give the probability of M given M_0.

Let u_k denote the summand in the right-hand side of (7). Then for $1 \le k < n$

$$\frac{u_{k+1}}{u_k} = \frac{n-k}{(k+1)(r-1)^s} \prod_{i=1}^{s-1} \frac{s(rn-2n+k)+i}{sn-sk-i}. \tag{8}$$

We first eliminate $k \le \epsilon n$ and $n-k \le \epsilon n$ from consideration, where $\epsilon = \epsilon(r,s)$ is small.

From (8), when $k \le n/(10r)$ we have $u_{k+1}/u_k \ge 5$. Hence

$$\begin{aligned} \sum_{k=0}^{\lfloor n/(20r) \rfloor} u_k &\le 2u_{\lfloor n/(20r) \rfloor} \\ &\le \frac{1}{5^{n/(20r)}} u_{\lfloor n/(10r) \rfloor}, \end{aligned}$$

and so the first $n/(20r)$ terms can be "ignored". Similarly, if for some $\epsilon > 0$ we have $k \ge n(1-\epsilon)$ then

$$\frac{u_{k+1}}{u_k} \ge \frac{(r-1-\epsilon)^{s-1}}{(r-1)^s \epsilon^{s-2}}. \tag{9}$$

Also $u_n = 1$ and since $\sum u_k \ge \mathbf{E}(Z)$ we can also ignore $k \ge n(1-r^{-s})$. Thus on applying Stirling's Formula and putting $k = n(1+x)/r$ we get

$$\begin{aligned} \frac{\mathbf{E}(Z^2)}{\mathbf{E}(Z)^2} &\approx \sum_x \frac{r}{\sqrt{2\pi(1+x)(r-1-x)n}} \left(\left(\frac{1}{1+x} \right)^{1+x} \right. \\ &\times \left. \left(1 + \frac{x}{(r-1)^2} \right)^{(s-1)((r-1)^2+x)} \left(1 - \frac{x}{r-1} \right)^{(s-2)(r-1-x)} \right)^{n/r} \\ &= \sum_x \frac{r}{\sqrt{2\pi(1+x)(r-1-x)n}} \left(\frac{1}{(1+x)^{1+x}} \exp \left\{ x + \right. \right. \\ &\left. \left. \sum_{k=2}^{\infty} \frac{x^k}{k(k-1)(r-1)^{k-1}} \left(s-2+\frac{(-1)^k(s-1)}{(r-1)^{k-1}} \right) \right\} \right)^{n/r}. \end{aligned} \tag{10}$$

The range of summation for x is $\{-1+\frac{rk}{n} : n/(20r) \le k \le n(1-r^{-s})\}$. Thus $-1 < x < r-1$. Note that the term with $x \approx 0$ corresponding to $k = \lfloor n/r \rfloor$ is approximately one and so we can eliminate any terms of order $o(n^{-2})$.

We continue with the terms with $|x| < 1$. Here we can expand $(1+x)^{1+x}$ and see that they contribute

$$\sum_{|x|<1} \frac{r}{\sqrt{2\pi(1+x)(r-1-x)n}} \exp\left\{\frac{n}{r}\left(\sum_{k=2}^{\infty} \frac{x^k}{k(k-1)}\left((-1)^{k-1} + \frac{s-2}{(r-1)^{k-1}} + \frac{(-1)^k(s-1)}{(r-1)^{2k-2}}\right)\right)\right\} \le \quad (11)$$

$$\sum_{|x|<1} \frac{r}{\sqrt{2\pi(1+x)(r-1-x)n}} \exp\left\{-\frac{n}{r}\left(\frac{r(r-s)}{2(r-1)^2}x^2 - \left(1+\frac{s-2}{(r-1)^2}\right)\frac{x^3}{6}\right)\right\} \le$$

$$\sum_{|x|<1} \frac{r}{\sqrt{2\pi(1+x)(r-1-x)n}} \exp\left\{-\frac{(r-s)n}{2(r-1)^2}x^2\right\}. \quad (12)$$

We will subsequently eliminate the terms with $x > 1$ as being insignificant and so from (10) and (12),

$$\begin{aligned}
\frac{\mathbf{E}(Z^2)}{\mathbf{E}(Z)^2} &\approx \frac{r}{\sqrt{2\pi(r-1)n}} \sum_{|x| \le \log n/\sqrt{n}} \exp\left\{-\frac{(r-s)n}{2(r-1)^2}x^2 + O((\log n)^3/\sqrt{n})\right\} \\
&\approx \frac{1}{\sqrt{2\pi(r-1)}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(r-s)}{2(r-1)^2}x^2\right\} dx \\
&= \sqrt{\frac{r-1}{r-s}},
\end{aligned}$$

as claimed. (Note that in going from the first line to the second line, the factor r disappears as x changes in steps of r/n.)

Now to deal with $x > 1$. Returning to (10) we bound from above its right-hand side, for $x > 1$, by

$$\sum_{x>1}\left(\frac{1}{(1+x)^{1+x}}\exp\left\{x+\sum_{k=2}^{\infty}\frac{x^k}{k(k-1)(r-1)^{k-1}}\left(s-2+\frac{s-1}{r-1}\right)\right\}\right)^{n/r} =$$

$$\sum_{x>1}\left(\frac{1}{(1+x)^{1+x}}\exp\left\{x+\left(\frac{s-2}{r-1}+\frac{s-1}{(r-1)^2}\right)x^2\sum_{k=2}^{\infty}\frac{x^{k-2}}{k(k-1)(r-1)^{k-2}}\right\}\right)^{n/r} \leq$$

$$\sum_{x>1}\left(\frac{1}{(1+x)^{1+x}}\exp\left\{x+\left(\frac{s-2}{r-1}+\frac{s-1}{(r-1)^2}\right)x^2\right\}\right)^{n/r}, \quad (13)$$

since $x < r-1$ in the summation.

Now consider

$$\phi(x) = \phi_{s,r}(x) = \log\left(\frac{1}{(1+x)^{1+x}}\exp\{x+\zeta x^2\}\right)$$

where $\zeta = \frac{s-2+\frac{s-1}{r-1}}{r-1}$.

$$\begin{aligned}\phi'(x) &= 2\zeta x - \log(1+x)\\ \phi''(x) &= 2\zeta - \frac{1}{1+x}\end{aligned}$$

Observe first that $2\zeta < \log 2$ for all $s \geq 3$ and $\sigma_r > s$. Also ϕ will be concave and decreasing until $x = \frac{1}{2\zeta} - 1$ and convex from then on. Also for fixed s and $x \geq 1$, $\phi(x)$ decreases with r. Our strategy is now as follows: taking $r = f(s)$ (see Table 1) we let $\epsilon = 1/7$ in (9) and put $x_s = \frac{6}{7}r - 1$. We then verify that

$$\frac{(r-(8/7))^{s-1}7^{s-2}}{(r-1)^s} \geq 1 \qquad \text{for } r \geq f(s) \quad (14)$$

and

$$\phi_{s,f(s)}(1), \phi_{s,f(s)}(x_s) \leq -.0001\ . \quad (15)$$

Then in the range $x \in [1, x_s]$ we can use (13) and (15) and in the range $[x_s, r-1]$ we can use (9) and (14) to show that the contribution of $x > 1$ is negligible.

We leave the detailed verification of (14) and (15) to the reader. ((14) is trivial, as is $\phi_{s,f(s)}(1) \leq -.0001$. The remaining inequality is a bit close for small s, but nevertheless true. For large s, $f(s) \approx e^{s-1}$ is a good approximation. Also, for $s \geq 4$ we can take $\epsilon = 1/5$ and $x_s = \frac{4}{5}r - 1$ which makes things easier.)

5 Cycles

First for $k > 2$,

$$\begin{aligned} \mathbf{E}(C_k) &\approx \binom{sn}{k}\frac{(k-1)!}{2}(r(r-1))^k \binom{srn}{s-2}^k \frac{(s!)^k (rn)^k}{(srn)^{sk}} \\ &\approx \frac{((s-1)(r-1))^k}{2k}. \end{aligned}$$

Explanation: $\binom{sn}{k}$ accounts for choosing the $v_1, v_2, \ldots, v_k$. $(k-1)!/2$ counts the cyclic orderings. $(r(r-1))^k$ counts the choices of points in the blocks W_{v_i}. $\binom{srn}{s-2}^k$ approximates the choices of the remaining $k(s-2)$ points. Then we have the probability that the k chosen s-tuples are in F.

When $k = 2$,

$$\begin{aligned} \mathbf{E}(C_2) &\approx \binom{sn}{2}\binom{r}{2}^2 2\binom{srn}{s-2}^2 \frac{(s!)^2 (rn)^2}{(srn)^{2s}} \\ &\approx \frac{(r-1)^2(s-1)^2}{4}, \end{aligned}$$

and when $k = 1$,

$$
\begin{aligned}
E(C_1) &\approx sn\binom{r}{2}\binom{srn}{s-2}\frac{s!rn}{(srn)^s} \\
&\approx \frac{(s-1)(r-1)}{2}
\end{aligned}
$$

Thus $\mathbf{E}(C_k) = \lambda_k$, for fixed $k \geq 1$. Routine calculations can strengthen this to show that C_k is asymptotically Poisson with this parameter and that in fact $C_1, C_2, \ldots, C_b$ are asymptotically independent. This *proves* Lemma 2.

6 Proof of Lemma 3

Let M_0 be some fixed perfect matching . Then

$$
\begin{aligned}
E_{\mathbf{c}} &= \sum_{F \in \Omega_{\mathbf{c}}} \frac{1}{|\Omega_{\mathbf{c}}|} \sum_{M \subseteq F} 1 \\
&= \sum_{M} \sum_{\substack{F \supseteq M \\ F \in \Omega_{\mathbf{c}}}} \frac{1}{|\Omega_{\mathbf{c}}|} \frac{|\Omega|}{|\Omega|} \\
&= \frac{|\Omega|}{|\Omega_{\mathbf{c}}|} \sum_{M} \mathbf{Pr}(F \supseteq M \text{ and } F \in \Omega_{\mathbf{c}}) \\
&= \frac{\mathbf{Pr}(F \supseteq M_0)}{\mathbf{Pr}(\Omega_{\mathbf{c}})} \sum_{M} \mathbf{Pr}(F \in \Omega_{\mathbf{c}} \mid F \supseteq M) \\
&= \frac{\mathbf{E}(Z)\mathbf{Pr}(F \in \Omega_{\mathbf{c}} \mid F \supseteq M_0)}{\mathbf{Pr}(\Omega_{\mathbf{c}})}. \qquad (16)
\end{aligned}
$$

Let E_t, $t = 0, 1, \ldots k_0 = \lfloor k/2 \rfloor$ denote the expected number of k-cycles which contain t s-tuples from M_0. Then $E_0 = ((s-1)(r-2))^k/(2k)$ and for $t \geq 1$

$$
E_t \approx \left[\binom{n}{t}\frac{(t-1)!}{2}(s(s-1))^t(r-1)^{2t}\binom{k-t-1}{t-1}\right]\left[\frac{(s!)^{k-t}((r-1)n)^{k-t}}{(s(r-1)n)^{s(k-t)}}\right]
$$

$$
\begin{aligned}
& \times \left[\binom{sn}{k-2t} (k-2t)!((r-1)(r-2))^{k-2t} \binom{s(r-1)n}{s-2}^{k-t} \right] \\
\approx \quad & ((s-1)(r-2))^k \frac{1}{2t} \binom{k-t-1}{t-1} \left(\frac{r-1}{(r-2)^2} \right)^t .
\end{aligned}
$$

Explanation: consider the first term inside [ ]'s. Choose t s-tuples T from M_0 and cyclically order them $\left(\binom{n}{t}\frac{(t-1)!}{2}\right)$. Choose ordered pairs of elements of tuples to *connect* with non-M_0 tuples $((s(s-1))^t)$. For each such point choose an element from the same block to go in a non-M_0 tuple $((r-1)^{2t})$. Choose $x_1, x_2, \ldots, x_t \geq 1$ where $x_1 + x_2 + \cdots x_t = k - t$. There will be x_i non-M_0 tuples between the i'th and $(i+1)$'th M_0 tuple $\left(\binom{k-t-1}{t-1}\right)$. Now consider the third term [ ]. We choose $k - 2t$ members U of V and order them $\left(\binom{sn}{k-2t}(k-2t)!\right)$. They are to be placed in s-tuples which will then be put *between* the tuples in T. Choose ordered pairs from each $W_u, u \in U$ $(((r-1)(r-2))^{k-2t})$. Then choose the remaining $(s-2)(k-t)$ points for the non-M_0 tuples $\left(\approx \binom{s(r-1)n}{s-2}^{k-t}\right)$. The middle term [ ] is simply the conditional probability that the cho sen tuples are in F.

Thus

$$
\mathbf{E}(C_k \mid M_0) = \frac{((s-1)(r-2))^k}{2k} + \frac{((s-1)(r-2))^k}{2} \sum_{t=1}^{k_0} \frac{\theta^t}{t} \binom{k-t-1}{t-1},
$$

where

$$
\theta = \frac{r-1}{(r-2)^2}.
$$

Now

$$
\begin{aligned}
\sum_{t=1}^{k_0} \frac{\theta^t}{t} \binom{k-t-1}{t-1} \quad &= \quad \theta^k \sum_{t=1}^{k_0} \frac{\theta^{t-k}}{k-t} \binom{k-t}{t} \\
&= \quad \theta^k [x^k] \sum_{t=1}^{k_0} \left(\frac{x(1+x)}{\theta} \right)^{k-t} \frac{1}{k-t}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{k} + \theta^k [x^k] \sum_{j=\lceil k/2 \rceil}^{k} \frac{1}{j} \left(\frac{x(1+x)}{\theta} \right)^j \\
&= -\frac{1}{k} + \theta^k [x^k] \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{x(1+x)}{\theta} \right)^j \\
&= -\frac{1}{k} - \theta^k [x^k] \log \left(1 - \frac{x(1+x)}{\theta} \right) \\
&= -\frac{1}{k} - \theta^k [x^k] \log \left[\left(1 + \frac{x}{\frac{1}{2} + \sqrt{\theta + \frac{1}{4}}} \right) \left(1 + \frac{x}{\frac{1}{2} - \sqrt{\theta + \frac{1}{4}}} \right) \right] \\
&= -\frac{1}{k} - \theta^k \frac{(-1)^{k-1}}{k} \left[\left(\frac{1}{\frac{1}{2} + \sqrt{\theta + \frac{1}{4}}} \right)^k + \left(\frac{1}{\frac{1}{2} - \sqrt{\theta + \frac{1}{4}}} \right)^k \right] \\
&= -\frac{1}{k} \left(1 + (-1)^{k-1} \left(\frac{r-1}{r-2} \right)^k \left(\frac{1}{(r-1)^k} + (-1)^k \right) \right).
\end{aligned}
$$

Thus, putting $\mu_k = \mathbf{E}(C_k \mid M_0)$ we see that

$$
\begin{aligned}
\mu_k &\approx \frac{((s-1)(r-1))^k}{2k} \left(1 + \frac{(-1)^k}{(r-1)^k} \right) \\
&= \lambda_k \left(1 + \frac{(-1)^k}{(r-1)^k} \right).
\end{aligned}
$$

Of course, further calculations will show that, given $F \supseteq M_0$, the C_k are asymptotically independently Poisson with means μ_k. Hence, from (16),

$$
E_{\mathbf{c}} \approx \mathbf{E}(Z) \prod_{k=1}^{b} \left(\frac{\mu_k}{\lambda_k} \right)^{c_k} e^{\lambda_k - \mu_k}. \tag{17}
$$

So,

$$
\begin{aligned}
\sum_{\mathbf{c} \in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 &\approx \mathbf{E}(Z)^2 \sum_{\mathbf{c} \in S(x)} \prod_{k=1}^{b} \left(\frac{\mu_k^2}{\lambda_k} \right)^{c_k} \frac{e^{-(2\mu_k - \lambda_k)}}{c_k!} \\
&= \mathbf{E}(Z)^2 \prod_{k=1}^{b} \sum_{c_k = \lambda_k - x\lambda_k^{2/3}}^{c_k = \lambda_k + x\lambda_k^{2/3}} \left(\frac{\mu_k^2}{\lambda_k} \right)^{c_k} \frac{e^{-(2\mu_k - \lambda_k)}}{c_k!} \qquad (18)
\end{aligned}
$$

We need to estimate

$$e^{-(\mu_k^2/\lambda_k)}\left(\sum_{c_k=0}^{\lambda_k-x\lambda_k^{2/3}}\left(\frac{\mu_k^2}{\lambda_k}\right)^{c_k}\frac{1}{c_k!}+\sum_{c_k=\lambda_k+x\lambda_k^{2/3}}^{\infty}\left(\frac{\mu_k^2}{\lambda_k}\right)^{c_k}\frac{1}{c_k!}\right). \tag{19}$$

First put

$$\begin{aligned}\lambda_k - x\lambda_k^{2/3} &= (1-\alpha_k)\left(\frac{\mu_k^2}{\lambda_k}\right),\\ \lambda_k + x\lambda_k^{2/3} &= (1+\beta_k)\left(\frac{\mu_k^2}{\lambda_k}\right)\end{aligned}$$

where $\alpha_k, \beta_k \geq \frac{x}{2\lambda_k^{1/3}}$ when x is sufficiently large.

From Alon and Spencer [1], p239 we obtain

$$\begin{aligned}\sum_{c_k=0}^{(1-\alpha_k)(\mu_k^2/\lambda_k)}\left(\frac{\mu_k^2}{\lambda_k}\right)\frac{e^{-(\mu_k^2/\lambda_k)}}{c_k!} &\leq e^{-\alpha_k^2\mu_k^2/(2\lambda_k)}\\ &\leq e^{-x^2\lambda_k^{1/3}/10},\end{aligned} \tag{20}$$

and

$$\begin{aligned}\sum_{c_k=(1+\beta_k)(\mu_k^2/\lambda_k)}^{\infty}\left(\frac{\mu_k^2}{\lambda_k}\right)\frac{e^{-(\mu_k^2/\lambda_k)}}{c_k!} &\leq \left(\frac{e^{\beta_k}}{(1+\beta_k)^{1+\beta_k}}\right)^{\mu_k^2/\lambda_k}\\ &\leq \left(\frac{\exp\{x/(2\lambda_k^{1/3})\}}{(1+(x/(2\lambda_k^{1/3})))^{1+(x/(2\lambda_k^{1/3}))}}\right)^{\lambda_k/2}\end{aligned} \tag{21}$$

If $x\lambda_1^{1/3} \geq 40\gamma$ then $x\lambda_k^{1/3} \geq 40\gamma$ for $k = 1, 2, \ldots, b$ and then the right-hand side of (20) is at most $e^{-4\gamma x}$ for $k = 1, 2, \ldots, b$.

On the other hand to make the right-hand side of (21) less than $e^{-4\gamma x}$ we need to make

$$\phi(x/(2\lambda_k^{1/3})) \geq 16\gamma/\lambda_k^{2/3}, \tag{22}$$

where

$$\phi(y) = \frac{1+y}{y}\log(1+y) - 1.$$

Now when $y \leq 1$ we have $\phi(y) \geq y/3$ and making $x \geq 96\gamma$ handles those k for which $48\gamma/\lambda_k^{2/3} \leq 1$. The set of k for which $48\gamma/\lambda_k^{1/3} > 1$ depends only on γ (i.e. is finite) and we can clearly increase x to make (22) true for all of these.

Hence, for x sufficiently large,

$$\sum_{\mathbf{c}\in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 \geq \mathbf{E}(Z)^2(1 - be^{-3\gamma x}) \prod_{k=1}^{b} \exp\left\{\frac{(\mu_k - \lambda_k)^2}{\lambda_k}\right\}. \qquad (23)$$

Also

$$\begin{aligned} \prod_{k=b+1}^{\infty} \exp\left\{\frac{(\mu_k - \lambda_k)^2}{\lambda_k}\right\} &= \exp\left\{\sum_{k=b+1}^{\infty} \frac{(s-1)^k}{2k(r-1)^k}\right\} \\ &\leq \left(1 - \left(\frac{s-1}{r-1}\right)^b\right)^{-1}. \end{aligned}$$

Thus, from (23), with

$$1 - \theta = \left(1 - be^{-3\alpha x}\right)\left(1 - \left(\frac{(s-1)}{r-1}\right)^b\right),$$

$$\begin{aligned} \sum_{\mathbf{c}\in S(x)} \pi_{\mathbf{c}} E_{\mathbf{c}}^2 &\geq (1-\theta)\mathbf{E}(Z)^2 \prod_{k=1}^{\infty} \exp\left\{\frac{(\mu_k - \lambda_k)^2}{\lambda_k}\right\} \\ &= (1-\theta)\mathbf{E}(Z)^2 \exp\left\{\frac{1}{2}\sum_{k=1}^{\infty} \frac{(s-1)^k}{k(r-1)^k}\right\} \\ &= (1-\theta)\mathbf{E}(Z)^2 \sqrt{\frac{r-1}{r-s}}. \end{aligned}$$

This completes the proof of Lemma 3.

7 Proof of Lemma 4

First we quote a lemma from [6].

Lemma 5 *Let $\eta_1, \eta_2, \ldots$ be given. Suppose that $\eta_1 > 0$ and that for some $c > 1, \eta_{i+1}/\eta_i > c$ for all $i > 1$. Then uniformly over $x \geq 1$,*

$$R(x) = \sum_{i=1}^{\infty} \sum_{t=\eta_i(1+y_i)}^{\infty} \frac{\eta_i^t}{t! e^{\eta_i}} = O(e^{-c_0 x})$$

where $y_i = x\eta_i^{-1/3}$ and $c_0 = \min\{\eta_1^{1/3}, \eta_1^{2/3}\}/4$.

(a) Putting $\eta_i = \lambda_i$ satisfies the conditions of Lemma 5 with $c = r - 1$. Now

$$\begin{aligned} \overline{\pi} &\leq \sum_{k=1}^{b} \sum_{c \geq \lambda_k(1+y_k)} \mathbf{Pr}(C_k = c) \\ &\approx \sum_{k=1}^{b} \sum_{c \geq \lambda_k(1+y_k)} \frac{{\lambda_k}^c e^{-\lambda_k}}{c!} \\ &= O(e^{-\alpha x}), \end{aligned}$$

for some constant α, independent of x.

(b) Applying (17) we obtain

$$\begin{aligned} E_{\mathbf{c}} &\approx \mathbf{E}(Z) \prod_{k=1}^{b} \left(1 + \frac{(-1)^k}{(r-1)^k}\right)^{c_k} \exp\left\{(-1)^{k-1}\frac{(s-1)^k}{2k}\right\} \\ &\geq AB^x, \end{aligned}$$

where

$$A = \prod_{k=1}^{b} \left(1 + \frac{(-1)^k}{(r-1)^k}\right)^{\lambda_k} \exp\left\{(-1)^{k-1}\frac{(s-1)^k}{2k}\right\}$$

and

$$B = \prod_{k \text{ odd}} \left(1 - \frac{1}{(r-1)^k}\right)^{\lambda_k^{2/3}} \prod_{k \text{ even}} \left(1 + \frac{1}{(r-1)^k}\right)^{-\lambda_k^{2/3}}.$$

Now

$$\begin{aligned} A &= \prod_{k=1}^{b} \exp\left\{(-1)^k \left(\frac{\lambda_k}{(r-1)^k} - \frac{\lambda_k}{2(r-1)^{2k}} + \frac{\lambda_k}{3(r-1)^{3k}} + \cdots\right) + (-1)^{k-1}\frac{(s-1)^k}{2k}\right\} \\ &\geq \prod_{k=1}^{\infty} \exp\left\{-\frac{(s-1)^k}{4k(r-1)^k}\right\} \\ &= \left(\frac{r-s}{r-1}\right)^{1/4}. \end{aligned}$$

Now

$$\begin{aligned} B &\geq \prod_{k=1}^{\infty} \left(1 - \frac{1}{(r-1)^k}\right)^{\lambda_k^{2/3}} \\ &\geq \exp\left\{-\sum_{k=1}^{\infty} \frac{\lambda_k^{2/3}}{(r-1)^k - 1}\right\}. \end{aligned}$$

The sum in the exponential term is convergent and so B is bounded below by a positive absolute constant.

References

[1] N.Alon and J.H.Spencer, *The Probabilistic Method*, John Wiley and Sons, New York 1992.

[2] E.A.Bender and E.R.Canfield, The asymptotic number of labelled graphs with given degree sequences, *Journal of Combinatorial Theory, Series A*, 24 (1978) 296-307.

[3] B.Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European Journal on Combinatorics*, 1 (1980) 311-316.

[4] P.Erdös and A.Rényi, *On the existence of a factor of degree one of a connected random graph*, Acta Math. Acad. Sci. Hungar. 17 (1966) 359-368.

[5] Łuczak and A.Ruciński, *Tree matchings in random graph processes*, SIAM Journal on Discrete Mathematics 4 (1991) 107-120.

[6] R.W.Robinson and N.C.Wormald, *Almost all cubic graphs are Hamiltonian*, Random Structures and Algorithms 3 (1992) 117-126.

[7] R.W.Robinson and N.C.Wormald, *Almost all regular graphs are Hamiltonian*, Random Structures and Algorithms to appear.

[8] A.Ruciński, *Matching and covering the vertices of a random graph by copies of a given graph*, Discrete Mathematics 105 (1992) 185-197.

[9] J.Schmidt-Pruzan and E.Shamir