$(k+1)$-cores have k-factors

Siu On Chan* Michael Molloy*

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Abstract

We prove that almost surely the first non-empty $(k+1)$-core to arise during the random graph process will have a k-factor. Thus the threshold for the appearance of a k-regular subgraph is at most the threshold for the $(k+1)$-core. This improves a result of Pralat, Verstraete and Wormald [4] and proves a conjecture of Bollobas, Kim and Verstraete [2].

1 Introduction

This paper concerns k-regular subgraphs of random graphs. A natural starting point for such a study is with the k-core; i.e. the unique maximal subgraph with minimum degree at least k. Pittel, Spencer and Wormald[3] determined the threshold $c_k \approx k + \sqrt{k \log k}$ for the appearance of a non-empty k-core in $G_{n,p=c/n}$[1]. So for $c < c_k$, a.s.[2] $G_{n,p=c/n}$ has no non-empty k-core and hence a.s. has no k-regular subgraph. In [2], Bollobás, Kim and Verstraete studied the threshold for the appearance of a 3-regular subgraph, and determined that it is strictly larger than c_3. They also conjectured that the threshold for a k-regular subgraph is strictly larger than c_k for all $k \geq 4$. Pretti and Weigt[5] used some statistical physics techniques to predict the opposite: for every $k \geq 4$, the threshold for the appearance of a k-regular subgraph is c_k. In other words, for every $c > c_k$. a.s. the k-core contains a k-regular subgraph. Those conflicting conjectures remain unresolved.

A k-factor of a graph G is a spanning k-regular subgraph; note that if G has a k-factor, then $k \times |V(G)|$ must be even. G is *k-factor-critical* if for every $v \in V(G)$, $G - v$ has a k-factor. Let C denote the first non-empty $(k+2)$-core to arise during the random graph process[3].

*Dept of Computer Science, University of Toronto, {siuon,molloy}@cs.toronto.edu.

[1]The random graph with n vertices where each of the $\binom{n}{2}$ possible edges appears independently with probability p.

[2]A property holds *almost surely (a.s.)* if it holds with probability tending to 1 as $\lim_{n\to\infty}$.

[3]The *random graph process* begins with n vertices and no edges, and then repeatedly adds an edge chosen uniformly at random from amongst those edges not yet present. $G_{n,M}$ is the graph obtained after M steps.

Pralat, Verstraete and Wormald[4] proved that if k is sufficiently large then a.s.: if $k \times |V(C)|$ is even then C contains a k-factor and if $k \times |V(C)|$ is odd then C is k-factor-critical. In this paper we extend that result to the $(k+1)$-core:

Theorem 1.1 *There is an absolute constant k_0 such that for all $k \geq k_0$, a.s. K, the first non-empty $(k+1)$-core to arise during the random graph process, satisfies:*

(a) if $k \times |V(K)|$ is even then K has a k-factor;

(b) if $k \times |V(K)|$ is odd then K is k-factor-critical.

This result is best possible (for large k) in that, as observed in [4], a.s. the first nonempty k-core to appear does not contain a k-factor, because it a.s. contains many vertices of degree greater than k whose neighbours all have degree exactly k.

Theorem 1.1 implies that a.s. if the random graph $G_{n,M}$ contains a $(k+1)$-core then that $(k+1)$-core contains a k-regular subgraph. This proves the first part of Conjecture 1.3 from [2]. It also establishes that the threshold for the appearance of a k-regular subgraph is at most the threshold for the appearance of a $(k+1)$-core in the $G_{n,M}$ model. By monotonicity of these two properties, it establishes the same for the $G_{n,p}$ model (see eg. Theorem 2.2 of [1] and the discussion following it). [2] remarked that perhaps a.s. the $(k+1)$-core of the random graph will contain a k-factor (so long as its size times k is even). Our arguments show that this is true for the random graph process for up to $O(\sqrt{k \log k})n$ steps after the first non-empty $(k+1)$-core appears and that it is true for $G_{n,p=c/n}$ when $c_{k+1} < c < c_{k+1} + O(\sqrt{k \log k})$.

Our proof is very close to that from [4], and we focus on their derivation from Tutte's theorem. Using $\omega(H)$ to denote the number of components of a subgraph H, and $\lambda(X, Y)$ to denote the number of edges with one endpoint in X and the other in Y, we prove:

Lemma 1.2 *If K is defined as in Theorem 1.1 then a.s.*

(a) every pair of disjoint sets $S, T \subset V(K)$ for which $S \cup T \neq \emptyset$ satisfies

$$\sum_{v \in T} d(v) + k|S| \geq \omega(K - (S \cup T)) + k|T| + \lambda(S, T); \tag{1}$$

(b) every pair of disjoint sets $S, T \subset V(K)$ for which $S \cup T \neq \emptyset$ and $|S| + \omega(K - (S \cup T)) \geq 2$ satisfies

$$\sum_{v \in T} d(v) + k|S| \geq \omega(K - (S \cup T)) + k|T| + \lambda(S, T) + 1. \tag{2}$$

The main argument from [4] was to prove Lemma 1.2, when K is the $(k+2)$-core. Their proof covered four separate cases for the sizes of S, T. Case 1 is straightforward - we require a different argument for the setting of this paper, but it is not difficult. Case 2 is where the main new ideas are required. Their arguments for Cases 3 and 4 apply to the setting of this paper, so we didn't need any new ideas there. The reader who is already familiar with [4] may want to skip directly to Case 2 (in particular, Case 2b).

We close this section by showing how Lemma 1.2 implies our main theorem.

Proof of Theorem 1.1: Lemma 2 of [4] shows that every multigraph K with $k \times |V(K)|$ even that satisfies the condition of Lemma 1.2(a) has a k-factor. So we turn to the case where $k \times |V(K)|$ is odd.

We use the following construction from [4]: Consider any $v \in V(K)$ and create K' by adding a new vertex v' with k parallel edges from v' to v. Thus $|V(K')|$ is even. We will show that K' satisfies the condition of Lemma 1.2(a) (upon replacing K by K') and thus has a k-factor. This implies that $K - v$ has a k-factor, as required.

Suppose that some $S', T' \subset V(K')$ with $S' \cup T' \neq \emptyset$ violate (1) (upon replacing S, T, K by S', T', K'). Let S, T be the sets obtained by removing v' from which, if either, of S', T' it belongs to. By checking each of the nine possible cases regarding whether v, v' are in S, T or neither it is straightforward to see that S, T violate (2). Furthermore, unless $v \in S$ and $v' \notin T$ S, T even violate (1). So it must be the case that either (i) $v \in S$ and $|S| + \omega(K - (S \cup T)) \leq 1$ or (ii) $S \cup T = \emptyset$.

In case (i) we must have $\omega(K - (S \cup T)) = 0$ and so $T = K - S$. It is well known that a.s. (i) the degree of v is less than $\log^2 n$ and (ii) the k-core has δn vertices of degree at least $k+2$, for some $\delta > 0$ (see eg. Lemma 6 from [4]). This implies that a.s. this choice of S, T cannot violate (2).

In case (ii) we must have $S \cup T = \{v'\}$. If $T' = \emptyset$ and $S' = \{v'\}$ then (1) holds. So the only remaining case is $S' = \emptyset$ and $T' = \{v'\}$. The same argument as in [4] shows that if this is the only pair $S', T' \in K'$ which violates (1) then K' has a k-factor, as required. $\square$

1.1 Preliminaries

We will make use of the following lemmas from [4] concerning the structure of K. (Actually, their lemmas were stated a bit differently in that they were in terms of the k-core and they worked in the $G_{n,p}$ model. But it is straightforward to adapt their proofs to obtain the statements below.)

Lemma 1.3 *(Lemma 3 of [4].) There is a positive constant γ such that a.s. for every set $X \subset V(K)$ of at most $\frac{1}{2}|V(K)|$ vertices, there are at least $\gamma(k+1)|X|$ edges from X to $K - X$.*

For the remainder of the paper, we use γ to denote the constant from Lemma 1.3.

Suppose that the first non-empty $(k+1)$-core arises when $M = 2cn$. The aforementioned result from [3] establishes that a.s. $c = c_{k+1} + o(1) \approx k + \sqrt{k \log k}$. We will often use the fact that $c < 2k$. We define $s(n) = \log n/(2ec \log\log n)$. Lemma 4 of [4] and the first step of its proof say:

Lemma 1.4 *(a) A.s. every subset Y of the vertices of $G_{n,M=2cn}$ with $|Y| \leq 4s(n)$ has at most $|Y|$ edges.*

(b) A.s. for every subset $Y \subseteq V(K)$ with $|Y| \leq s(n)$, $K - Y$ contains a component with more than $|V(K)| - 2s(n)$ vertices.

Remark: Actually, they prove part (a) for $|Y| \leq 2s(n)$. But the same proof works for $|Y| \leq 4s(n)$.

Finally, we often use the following bound which follows easily from Stirling's Inequality:

$$\binom{a}{b} \leq \left(\frac{ea}{b}\right)^b.$$

2 Proof of Lemma 1.2

As stated earlier, we focus on the same cases for $|S|, |T|$ as in [4].

Case 1: $|S| + |T| \leq s(n)$.

The proof of this case is similar to that from [4]. Let $\omega(K - (S \cup T)) = \ell + 1$. By Lemma 1.4(b), a.s. the sizes of $C_1, ..., C_\ell$, the ℓ smallest components of $K - (S \cup T)$, total less than $2s(n)$. So Lemma 1.4(a) implies that a.s. the subgraph X induced by $S \cup T \cup C_1 \cup ... \cup C_\ell$ has no more edges than vertices. Let X' be the graph obtained by contracting each C_i into the single vertex c_i. Since C_i has at most one cycle (by Lemma 1.4(a)) and every vertex of C_i has degree at least $k+1$ in X, it follows that $\deg(c_i) \geq k+1$. Since each c_i is only adjacent to vertices in $S \cup T$ we have $|E(X')| \geq (k+1)\ell + \lambda(S,T)$. Since X has no more edges than vertices and since each C_i is connected, $|E(X')| \leq |V(X')| = |S| + |T| + \ell$. Therefore:

$$|T| + k|S| \geq k\ell + \lambda(S,T) + (k-1)|S| = \omega(K - (S \cup T)) + \lambda(S,T) + (k-1)(|S| + \ell) - 1.$$

Since every vertex in T has degree at least $k+1$, this implies (1) and (2) for $|S| + \ell \geq 1$. If $|S| = \ell = 0$ and $S \cup T \neq \emptyset$ then we must have $|T| \geq 1, \omega(K - (S \cup T)) = 1$ and $\lambda(S,T) = 0$, and so (1) holds. This proves Lemma 1.2 for $|S| + |T| \leq s(n)$.

We fix an absolute constant ϵ_0, independent of k, chosen so that $\epsilon_0 < \frac{\gamma^2}{10^5}$ (recall γ from Lemma 1.3). Our next case is:

Case 2: $s(n) \leq |S| + |T| \leq \epsilon_0 n$

Many times, we will work with $G_{n,p=c/n}$ rather than $G_{n,M=cn}$. When we do so, the properties that we prove hold a.s. are monotone; i.e. they are preserved either under the addition of edges or under the deletion of edges. So it follows from Theorem 2.2 of [1] that the results translate from $G_{n,p=c/n}$ to $G_{n,M=cn}$. In each such proof, we show that the property holds a.s. for all $c < 2k$ and so we do not have to condition on $M = 2cn$ being the exact point that the first non-empty $(k+1)$-core arises in the random graph process.

Case 2a: $|T| \leq 20k|S|$.

The analysis for this case is very much like that of [4]. We first show that a.s. for every pair S, T with $|S| + |T| \leq \epsilon_0 n$, we have:

$$\omega(K - S \cup T) < \frac{1}{100}(|S| + |T|). \tag{3}$$

Proof of (3): Otherwise, let X be the set of vertices in all components of $K - S \cup T$ of size at most $n/2$. Since at most one component of $K - S \cup T$ has size greater than $n/2$, we have $|X| \geq \omega(K - S \cup T) - 1 > \frac{|S|+|T|}{200}$ (as S, T violate (3) and $|S| + |T| \geq s(n)$). By applying Lemma 1.3 to each component of X, we have $\lambda(X, S \cup T) \geq \gamma(k+1)|X|$. Let $Y = S \cup T$.

We will prove that a.s. $G_{n,p=c/n}$ contains no sets X, Y with $|Y| \leq \epsilon_0 n$, $|X| \geq |Y|$ and $\lambda(X, Y) \geq \gamma(k+1)|X|$. Let $xn = |X|$, and $yn = |Y|$. For any fixed x, y, the expected number of such sets X, Y is at most:

$$\begin{aligned}
& \binom{n}{yn}\binom{n}{xn}\binom{(yn)(xn)}{\gamma(k+1)xn}\left(\frac{c}{n}\right)^{\gamma(k+1)xn} \\
< \; & \left(\frac{e}{y}\right)^{yn}\left(\frac{e}{x}\right)^{xn}\left(\frac{exyn^2c}{\gamma(k+1)xn^2}\right)^{\gamma(k+1)xn} \\
< \; & \left(\frac{e}{y/200}\right)^{201xn}\left(\frac{2ey}{\gamma}\right)^{\gamma(k+1)xn} \qquad \text{since } x > \frac{y}{200} \text{ and } c < 2k \\
< \; & \left(\frac{800e^3y}{\gamma^2}\right)^{\frac{1}{2}\gamma(k+1)xn} \qquad \text{if } k \text{ satisfies } 201 < \frac{1}{2}\gamma(k+1) \\
< \; & \left(\frac{1}{2}\right)^{xn} \qquad \text{since } y \leq \epsilon_0 < \frac{1}{2}\left(\frac{\gamma^2}{800e^3}\right) \text{ and } \frac{1}{2}\gamma(k+1) > 1.
\end{aligned}$$

For each fixed x, there are at most $100xn$ choices for y, since $s(n) < |Y| \leq 100(|X| + 1)$. Therefore, summing over all x, y we find that the expected number of pairs X, Y with $|X| \geq \log n$ that violate (3) is less than:

$$\sum_{|X| \geq \log n} 100|X|\left(\frac{1}{2}\right)^{|X|} = o(1).$$

For $|X| < \log n$ we have $|Y| < 100(\log n + 1)$; i.e. $y < \frac{100(\log n+1)}{n}$. Thus $\left(\frac{800e^2y}{\gamma^2}\right)^{\frac{1}{2}\gamma(k+1)xn} < \frac{1}{n^3}$ (since $xn = |X| \geq 1$). There are fewer than n^2 choices for x, y and so the expected number of pairs X, Y with $|X| < \log n$ that violate (3) is $o(1)$. □

Next we show that a.s. for every pair of sets S, T with $s(n) \leq |S| + |T| \leq \epsilon_0 n$ we have:

$$\lambda(S,T) < \frac{101}{100}|T| + \frac{k}{2}|S|. \tag{4}$$

Proof of (4): Let $\sigma n = |S|$ and $\tau n = |T|$. For any choice of σ, τ, the expected number of such sets S, T in $G_{n,p=c/n}$ violating (4) is at most:

$$\begin{aligned}
\binom{n}{\sigma n}\binom{n}{\tau n}\binom{(\sigma n)(\tau n)}{\frac{101}{100}\tau n + \frac{k}{2}\sigma n}\left(\frac{c}{n}\right)^{\frac{101}{100}\tau n + \frac{k}{2}\sigma n} &< \left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{e\sigma\tau n^2 c}{(\frac{101}{100}\tau n + \frac{k}{2}\sigma n)n}\right)^{\frac{101}{100}\tau n + \frac{k}{2}\sigma n} \\
&= \left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{e\sigma\tau c}{\frac{101}{100}\tau + \frac{k}{2}\sigma}\right)^{\frac{101}{100}\tau n + \frac{k}{2}\sigma n}.
\end{aligned}$$

Note that

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau + \frac{k}{2}\sigma} < \frac{e\sigma\tau c}{\frac{k}{2}\sigma} < 4e\tau < \left(\frac{\tau}{2e}\right)^{\frac{100}{101}},$$

since $c < 2k$ and $\tau < \epsilon_0 < (16e^3)^{-100}$. Furthermore, if $\sigma > e^{-k/3}$ then

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau + \frac{k}{2}\sigma} < \left(\frac{\tau}{2e}\right)^{\frac{100}{101}} < e^{-1} < \left(\frac{\sigma}{2e}\right)^{\frac{2}{k}},$$

while if $\sigma \leq e^{-k/3}$ then for k sufficiently large we have:

$$\frac{e\sigma\tau c}{\frac{101}{100}\tau + \frac{k}{2}\sigma} < \frac{e\sigma\tau c}{\frac{101}{100}\tau} < ec\sigma^{1/2}\sigma^{1/2} < e(2k)e^{-k/6}\sigma^{1/2} < \sigma^{1/2} < \left(\frac{\sigma}{2e}\right)^{\frac{2}{k}}.$$

This implies that the expected number of pairs S, T with $|S| = \sigma n, |T| = \tau n$ is at most

$$\left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{\tau}\right)^{\tau n}\left(\frac{\tau}{2e}\right)^{\frac{100}{101}\frac{101}{100}\tau n}\left(\frac{\sigma}{2e}\right)^{\frac{2}{k}\frac{k}{2}\sigma n} = \left(\frac{1}{2}\right)^{(\sigma+\tau)n}.$$

For each choice of $y = |S| + |T|$, there are y choices for $|S|, |T|$. So the expected number of sets S, T violating (4) is at most:

$$\sum_{y=s(n)}^{n} y(\frac{1}{2})^y = o(1).$$

□

By (3) and (4) we have that a.s. every pair S, T with $s(n) \leq |S| + |T| \leq \epsilon_0 n$ and $|T| \leq 20k|S|$ satisfies:

$$\omega(K - S \cup T) + \lambda(S,T) < \frac{1}{100}(|T| + |S|) + \frac{101}{100}|T| + \frac{k}{2}|S| = \frac{102}{100}|T| + (\frac{k}{2} + \frac{1}{100})|S| < |T| + k|S| - 1,$$

where the last inequality uses $|T| \leq 20k|S|$. This implies (1) and (2) for Case 2a.

Case 2b: $|T| > 20k|S|$.

This case contains the main new ideas for this paper. As in Case 2a, we wish to do a first moment calculation which bounds $\lambda(S,T)$ and $\lambda(X, S\cup T)$. However, the possibility of $|T|$ being much higher than $|S|$ creates a combinatorial blowup in our first moment calculations. In particular, the $\binom{n}{|T|}$ term is far too big. So instead, we will bound $\lambda(S, N(S))$ and $\lambda(X,(S\cup N(S))\backslash X)$. The advantage for the first moment argument is that the choice of S determines $N(S)$ and so we don't need to include a large term counting the number of choices for $N(S)$.

The bound on $\lambda(S,N(S))$ will yield a bound on $\lambda(S,T)$ after observing that $\lambda(S,T) = \lambda(S,T\cap N(S))$. The bound on $\lambda(X,(S\cup N(S))\backslash X)$ will yield a bound on $\lambda(X,S\cup(T\cap N(S)))$. We will obtain a separate bound on $\lambda(X,T')$ for $T' = T - N(S)$; for that bound, the $\binom{n}{|T'|}$ term is not problematic. Indeed, calculations nearly identical to those for (3) yield that a.s. for every pair of sets X, T' with $|X| \geq \frac{1}{200}|T'|$ and $|T'| \leq \epsilon_0 n$ we have:

$$\lambda(X,T') < \frac{1}{2}\gamma k|X|. \tag{5}$$

We omit the repetitive details of the proof of (5).

Note that in Case 2b, we have $|S| + 20k|S| \leq |S| + |T| \leq \epsilon_0 n$ and so $|S| < \frac{\epsilon_0}{20k}n$. This allows us to make use of the following technical bound:

Lemma 2.1 *A.s. every set S in $G_{n,p=c/n}$ of size at most $\frac{\epsilon_0}{20k}n$ satisfies:*

(a) $\left(\frac{25|S\cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k}\frac{n^2}{|S|^2} < \frac{1}{2e^2}$.

(b) $\left(\frac{25|N(S)|}{n}\right)^{k/4}\frac{n}{|S|} < \frac{1}{2e}$.

Proof Note that (a) implies that $\frac{25|S\cup N(S)|}{\gamma n} < 1$ and so (a) implies (b). So we will prove (a). Let S^* be the $|S|$ vertices of largest degree in $G_{n,p}$ and let D be the sum of their degrees. Clearly $|N(S)| \leq D$. For $i \geq 0$, the expected number of vertices of degree i in $G_{n,p=c/n}$ is $\left(\frac{c^i}{i!}e^{-c} + o(1)\right)n$. Standard methods (eg Lemma 3.10 of [1]) show that this number is highly concentrated: A.s. for all i such that $\frac{c^i}{i!}n \geq \sqrt{n}$ we have (a) at most $\frac{c^i}{i!}n$ vertices have degree i and (b) at most $\sum_{j\geq i}\frac{c^j}{j!}n$ vertices have degree at least i. Also, it is well-known that the maximum degree in $G_{n,p=c/n}$ is a.s. less than $\log n$ (see eg. Exercise 3.5 of [1]). We will assume that these almost sure properties hold, and show that for every choice of $|S|$, the bound in (a) holds. This establishes our lemma.

Case 1: $|S| \leq n^{2/3}$. Since the maximum degree is less than $\log n$, we have $D \leq |S| \log n$ and so

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} \left(\frac{25|S|(\log n + 1)}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2}.$$

That product clearly increases with $|S|$ and so is at most:

$$\left(\frac{25n^{2/3}(\log n + 1)}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{(n^{2/3})^2} = o(1).$$

For the next two cases, we define I to be the largest integer such that $\frac{c^I}{I!}n \geq |S|$, and i^* to be the largest integer for which $\frac{c^{i^*}}{i^*!}n \geq \sqrt{n}$. It is easily verifed that $\sum_{i>i^*} \frac{c^i}{i!}n < 2\frac{c^{i^*+1}}{(i^*+1)!}n < 2\sqrt{n}$. Thus:

$$D < \sum_{i \geq I} i\frac{c^i}{i!}n + \sum_{i>i^*} \log n \times \frac{c^i}{i!}n < \sum_{i \geq I} i\frac{c^i}{i!}n + 2\sqrt{n}\log n.$$

Case 2: $|S| > n^{2/3}$ *and* $I \geq 4c$. Since $I \geq 4c$, it is easily verified that $\frac{c^I}{I!}n + \sum_{i \geq I} i\frac{c^i}{i!}n < 2I\frac{c^I}{I!}n$. Also, $I\frac{c^I}{I!}n > |S| > n^{2/3} > 2\sqrt{n}\log n$. So our bound above yields $|S| + D < 2I\frac{c^I}{I!}n + 2\sqrt{n}\log n < 3I\frac{c^I}{I!}n < 3I^2\frac{c^{I+1}}{(I+1)!}n < 3I^2|S|$. Thus:

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} < \left(\frac{25 \times 3I^2}{\gamma}\right)^2 \left(\frac{25 \times 3I\frac{c^I}{I!}}{\gamma}\right)^{\frac{1}{2}\gamma k - 2}.$$

This product is easily seen to decrease as I increases, and so it is at most

$$\left(\frac{25 \times 48c^2}{\gamma}\right)^2 \left(\frac{25 \times 12c\frac{c^{4c}}{(4c)!}}{\gamma}\right)^{\frac{1}{2}\gamma k - 2} < \frac{1}{2e^2},$$

for k (and hence c) sufficiently large.

Case 3: $I < 4c$. $D/|S|$ is monotone decreasing as $|S|$ increases, since increasing $|S|$ adds to S^* vertices of degree at most that of all those already in S^*. Therefore, $|S \cup N(S)|/|S|$ is at most the value $(|S| + D)/|S|$ at $I = 4c$ and $|S| = \frac{c^I}{I!}n$. Applying the arguments from Case 2, at that value of $|S|$ we have $|S| + D \leq 3I\frac{c^I}{I!}n = 3I|S| = 12c|S|$. Therefore, using the facts that $c < 2k$ and $|S| < \frac{\epsilon_0}{20k}n < \frac{\epsilon_0}{10c}n < \frac{\gamma}{25 \times 24c}n$ we have:

$$\left(\frac{25|S \cup N(S)|}{\gamma n}\right)^{\frac{1}{2}\gamma k} \frac{n^2}{|S|^2} < \left(\frac{25 \times 12c}{\gamma}\right)^2 \left(\frac{25 \times 12c|S|}{\gamma n}\right)^{\frac{1}{2}\gamma k - 2} < \left(\frac{25 \times 24k}{\gamma}\right)^2 \left(\frac{1}{2}\right)^{\frac{1}{2}\gamma k - 2} < \frac{1}{2e^2},$$

for k sufficiently large. $\square$

Next we bound $\lambda(X,(S\cup N(S))\backslash X)$. We will prove that a.s. for every pair of disjoint sets $S,X\subset V(K)$ with $|S|\leq\frac{\epsilon_0}{20k}n$ and $|X|\geq|S|$ we have:

$$\lambda(X,(S\cup N(S))\backslash X)<\frac{1}{2}\gamma k|X|. \tag{6}$$

Proof of (6): We show that there are a.s. no such sets violating (6) in $G_{n,p=c/n}$. We fix $|S|=\sigma n\leq\frac{\epsilon_0}{20k}n$, $|X|=xn\geq\sigma n$ and we let ρ be the solution to $\left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma k}\frac{1}{\sigma^2}=\frac{1}{2e^2}$. We invoke an arbitrary ordering on the vertices of $G_{n,p}$ and let $\lambda'(X,Y)$ denote the number of edges from X to the first $\min(|Y|,\rho n)$ vertices of Y. By Lemma 2.1(a), a.s. $\lambda'(X,(S\cup N(S))\backslash X)=\lambda(X,(S\cup N(S))\backslash X)$ for every choice of X,S. To compute the expected number of pairs S,X with $\lambda'(X,(S\cup N(S))\backslash X)\geq\frac{1}{2}\gamma k|X|$, we first choose S,X, then expose $N(S)$ and then bound the probability that $\lambda'(X,(S\cup N(S))\backslash X)\geq\frac{1}{2}\gamma k|X|$. Thus the expected number is at most:

$$\begin{aligned}
&\binom{n}{\sigma n}\binom{n}{xn}\binom{(\rho n)(xn)}{\frac{1}{2}\gamma kxn}\left(\frac{c}{n}\right)^{\frac{1}{2}\gamma kxn}\\
<\ &\left(\frac{e}{\sigma}\right)^{\sigma n}\left(\frac{e}{x}\right)^{xn}\left(\frac{ec\rho}{\frac{1}{2}\gamma k}\right)^{\frac{1}{2}\gamma kxn}\\
<\ &\left(\frac{e}{\sigma}\right)^{2xn}\left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma kxn} \qquad\qquad \text{since } \sigma<x \text{ and } c<2k\\
=\ &\left(\frac{1}{2}\right)^{xn} \qquad \text{since } \left(\frac{25\rho}{\gamma}\right)^{\frac{1}{2}\gamma k}=\frac{\sigma^2}{2e^2}.
\end{aligned}$$

The sum of $\left(\frac{1}{2}\right)^{xn}$ over all $|S|,|X|$ with $|X|\geq\max(|S|,\sqrt{\log n})$ is $o(1)$. For the case where $|S|\leq|X|<\sqrt{\log n}$, let H be the subgraph induced by X,S and the endpoints in $S\cup N(S)$ of $\frac{1}{2}\gamma k|X|$ edges from X. It is straightforward to show that H has more edges than vertices: Indeed, if ℓ is the number of vertices of H in $N(S)$, then $|E(H)|\geq\ell+\frac{1}{2}\gamma k|X|>\ell+2|X|$ and $|V(H)|=\ell+|S|+|X|\leq\ell+2|X|$. But H has at most $|S|+|X|+\frac{1}{2}\gamma k|X|=O(\sqrt{\log n})$ vertices and thus contradicts Lemma 1.4(a). This proves (6). □

Our final bound is that a.s. for every set $S\subset V(K)$ with $|S|\leq\frac{\epsilon_0}{20k}n$ we have:

$$\lambda(S,N(S))\leq|N(S)|+\frac{k}{4}|S|. \tag{7}$$

Proof of (7): Again, we prove that (7) a.s. holds for every such S in $G_{n,p}$. Consider any set S of σn vertices, and expose $N(S)$ without exposing the edges from S to $N(S)$. In particular, for each $v\notin S$, we let $d_S(v)$ denote the number of neighbours v has in S, and we expose whether $d_S(v)>0$. The random variables $d_S(v)$ are independent and their distributions are asymptotic to a Poisson variable with mean $c\sigma$. So the distribution of $d_S(v)$, conditioned on the event that $v \in N(S)$, is asymptotic to a truncated Poisson variable, which we denote by $Z_{\geq 1}(c\sigma)$. We use $Z(c\sigma)$ to denote a Poisson variable with mean $c\sigma$. For every $i > 1$, it is straightforward to verify that $\mathbf{Pr}(Z_{\geq 1}(c\sigma) = i+1) < \mathbf{Pr}(Z(c\sigma) = i)$. It follows that the probability that $\lambda(S, N(S)) > |N(S)| + \frac{k}{4}|S|$ is less than the probability that $|N(S)|$ copies of $Z(c\sigma)$ sum to at least $\frac{k}{4}|S|$; i.e. $\mathbf{Pr}(Z(|N(S)|c\sigma) \geq \frac{k}{4}|S|)$.

Let ν be the solution to $(25\nu)^{k/4}/\sigma = \frac{1}{2e}$. By Lemma 2.1(b), a.s. $|N(S)| < \nu n$. Note that for sufficiently large k, $\nu \leq \frac{1}{24}$ and so $\frac{k}{4}\sigma n > 2k\sigma\nu n > c\sigma\nu n$. It easily follows that $\mathbf{Pr}(Z(|N(S)|c\sigma) \geq \frac{k}{4}|S|) < 2\mathbf{Pr}(Z(|N(S)|c\sigma) = \frac{k}{4}|S|)$. So the expected number of sets S of size σn that violate (7) and for which $|N(S)| < \nu n$ is at most:

$$
\begin{aligned}
& \binom{n}{\sigma n} \times 2\frac{(c\sigma\nu n)^{\frac{k}{4}\sigma n}}{(\frac{k}{4}\sigma n)!} e^{-c\sigma\nu n} \\
< \;& 2\left(\frac{e}{\sigma}\right)^{\sigma n} \left(\frac{ec\sigma\nu n}{\frac{k}{4}\sigma n}\right)^{\frac{k}{4}\sigma n} \\
< \;& \left(\frac{e}{\sigma}\right)^{\sigma n} (25\nu)^{\frac{k}{4}\sigma n} \qquad \text{since } c < 2k \\
= \;& \left(\frac{1}{2}\right)^{\sigma n}.
\end{aligned}
$$

The sum of $\left(\frac{1}{2}\right)^{|S|}$ over all $|S| \geq \log n$ is $o(1)$. For the case where $|S| < \log n$, we use the easily verified fact that a.s. the maximum degree in $G_{n,p=c/n}$ is less than $\log n$ (see eg. Exercise 3.5 in [1]). Thus, replacing νn above by $|S| \log n$ we obtain a smaller bound of

$$
\left(\frac{en}{|S|}\right)^{|S|} \left(\frac{25|S|\log n}{n}\right)^{\frac{k}{4}|S|} < \frac{1}{n^2}.
$$

Multiplying this by the $\log n$ choices for $|S| < \log n$ yields a total of $o(1)$. □

Now consider any pair of sets S, T with sizes as in Case 2b. As in Case 2a, we let X be the set of vertices in all components of $K - S \cup T$ of size at most $n/2$ and we have $|X| \geq \omega(K - S \cup T) - 1$. If $|X| \geq \max(\frac{1}{200}|T \backslash N(S)|, |S|)$ then (5) and (6) imply:

$$
\lambda(X, S \cup T) = \lambda(X, T \backslash N(S)) + \lambda(X, S \cup (T \cap N(S))) \leq \lambda(X, T \backslash N(S)) + \lambda(X, (S \cup N(S)) \backslash X) < \gamma k |X|,
$$

which contradicts Lemma 1.3. Thus $|X| < \max(\frac{1}{200}|T \backslash N(S)|, |S|) \leq |S| + \frac{1}{200}|T \backslash N(S)|$. Therefore $\omega(K - S \cup T) \leq |X| + 1 \leq |S| + \frac{1}{200}|T \backslash N(S)|$.

$$
\lambda(S, N(S)) = \sum_{u \in N(S)} d_S(u) \geq |N(S) \backslash T| + \sum_{u \in N(S) \cap T} d_S(u) = |N(S) \backslash T| + \lambda(S, T).
$$

Thus (7) implies $\lambda(S,T) \leq \frac{k}{4}|S| + |T \cap N(S)|$. Therefore

$$\omega(K - S \cup T) + \lambda(S,T) \leq |S| + \frac{1}{200}|T \backslash N(S)| + \frac{k}{4}|S| + |T \cap N(S)| \leq k|S| + |T| - (\frac{3k}{4} - 1)|S|.$$

This implies (1) and also implies (2) if $|S| \geq 1$.

Suppose $|S| = 0$. Then $|T| \geq s(n)$ and $T = T \backslash N(S)$. Since we showed $\omega(K - S \cup T) \leq |X| + 1 \leq |S| + \frac{1}{200}|T \backslash N(S)| = \frac{1}{200}|T|$, we have

$$\omega(K - S \cup T) + \lambda(S,T) \leq \frac{1}{200}|T| < k|S| + |T| - 1.$$

This proves (2).

Case 3: $|S| + |T| \geq \epsilon_0 n$

This is covered by Cases 3 and 4 from [4]. The proofs from that paper also apply to the setting of this paper (after a straightforward adjustment of some of the constants).

In particular, if $|T| < \frac{1}{2}\epsilon_0 n$ then $|S| > \frac{1}{2}\epsilon_0 n$. The same analysis as in Case 3 of [4] shows that a.s. every such S, T satisfies $\lambda(S,T) \leq \frac{3}{4}k|S|$. Since $\sum_{v \in T} d(v) \geq (k+1)|T|$ and $\omega(G - (S \cup T)) < n < \frac{1}{4}k|S| - 1$ for $k > \frac{8}{\epsilon_0}$, (1) and (2) both hold.

If $|T| \geq \frac{1}{2}\epsilon_0 n$ then (10) from [4] states that there exists $\epsilon > 0$ such that a.s. $\lambda(S,T) \leq k|S| + (1-\epsilon)\sqrt{k \log k}|T|$ for every such S, T. (The proof of (10) does not rely on the fact that they are dealing with the $(k+2)$-core rather than the $(k+1)$-core.) The statement preceding (10) in [4] (after replacing ϵ by $\frac{\epsilon}{2}$) implies that for k sufficiently large, we a.s. have $\sum_{v \in T} d(v) > (k + (1 - \frac{\epsilon}{2}\sqrt{k \log k}))|T|$ for every such T. Since $\omega(G - (S \cup T)) + 1 < n < \frac{\epsilon}{2}\sqrt{k \log k}|T| - 1$ for $k > 4/(\epsilon\epsilon_0)^2$, this yields (1) and (2).

Remark: It is in this final step that we require $c \leq k + O(\sqrt{k \log k})$.

References

[1] B. Bollobás, *Random Graphs.* 2nd edition. Cambridge University Press, 2001.

[2] B. Bollobás, J.H. Kim and J. Verstraete, *Regular subgraphs of random graphs.* Rand. Struc. & Alg. **29** (2006), 1 - 13.

[3] B. Pittel, J. Spencer and N. Wormald, *Sudden emergence of a giant k-core in a random graph.* J.Comb.Th.(B) **67** (1996), 111 - 151.

[4] P. Pralat, J. Verstraete and N. Wormald, *On the threshold for k-regular subgraphs of random graphs.* Submitted.

[5] M. Pretti and M. Weigt, *Sudden emergence of q-regular subgraphs in random graphs.* Europhys. Lett. 75, 8 (2006).