THE DOMINATING NUMBER OF A RANDOM CUBIC GRAPH

Michael Molloy*
Department of Mathematics
Carnegie Mellon University
Pittsburgh PA15213, U.S.A.

Bruce Reed
Equipe Combinatoire
CNRS
Université Pierre et Marie Curie
Paris, France

November 2, 2000

Abstract

We show that if G is a random 3-regular graph on n vertices, then its dominating number, $D(G)$, almost surely satisfies $.2636n \leq D(G) \leq .3126n$.

In this paper, we consider the dominating number of a random 3-regular graph. A *dominating set* of a graph G, is a subset $S \subset V(G)$, such that each $v \in V(G) \backslash S$ is adjacent to some vertex in S. The *dominating number* of G, $D(G)$ is the size of the smallest dominating set of G. Because we are dealing with 3-regular graphs on n vertices, we will assume throughout this paper that n is even.

Trivially, if G is 3-regular, then $D(G) \geq \frac{1}{4}|V(G)|$, as each member of S is adjacent to at most 3 members of $V(G) \backslash S$. In [6], the second author proved a conjecture of Koch and Perles [1] that $D(G) \leq \frac{3}{8}|V(G)|$. Both of these bounds are tight.

Consider the probability space $\Omega_{G,n}$ whose domain is Υ_n, the set of all 3-regular graphs with vertex set $[n]$, and which has the uniform distribution. Recently, Robinson and Wormald

*Supported in part by NSF grant CCR9225008.

[7] showed that if G is chosen from $\Omega_{G,n}$, then G is almost surely (a.s.) Hamiltonian. This immediately implies that a.s. $D(G) \leq \lceil \frac{n}{3} \rceil$, as one can form a dominating set by proceeding around a Hamilton cycle, H of G and choosing every third vertex.

We consider a random 3-regular multigraph F, formed by taking the union of a uniformly selected random Hamilton cycle, H, on n vertices and a uniformly selected random perfect matching M on the same vertex set. More formally, consider the probability space $\Omega_{F,n}$, whose domain is Σ_n, the set of all pairs (H, M) where H is a Hamilton cycle and M is a perfect matching, both with vertex set $[n]$, and which has the uniform distribution. We now choose a random 3-regular multigraph F by selecting a member of $\Omega_{F,n}$ and setting $F = H \cup M$.

Note that F is *not* equivalent to G. For one thing, F is not guaranteed to be simple. Moreover, even if we chose (H, M) conditional on their edge sets not intersecting, graphs with many Hamilton cycles are more likely to be selected as F. However, it follows from the results of [7] that the distribution from which F is selected is sufficently close to that of $\Omega_{G,n}$ that if a property holds a.s. for F then it holds a.s. for G.

To us, this corollary of Robinson and Wormald's work is much more significant than the simple statement that a.e. cubic graph is Hamiltonian. In this paper, we show how to apply it to obtain results on the dominating number of random cubic graphs. We expect that the corollary will have many other applications in the study of random cubic graphs.

We will show that we can a.s. find a dominating set, S, in F of size much less than $\frac{n}{3}$. To do so, we simply modify the trivial procedure for finding a dominating set of size $\lceil \frac{n}{3} \rceil$ to take advantage of the edges of M. In particular, in the first procedure we travelled around H, adding every third vertex on H to S, the reasoning being that if s, t, u and v are consecutive vertices and we add s to S, then we can skip t as it will be covered by s and we can also skip u, as long as we cover it by picking v. If we use the edges of M to our advantage, then we may be able to skip u *without* being forced to pick v, in particular if it turns out that u is already covered by a member of S via M. It turns out that the edges of M help us substantially, and we get a.s. get a dominating set of size $(.3126 + o(1))n$.

We will also use an expected number argument to show that a.s. $D(G) \geq .2636n$, thus improving the trivial bounds on $D(G)$.

We also show that the dominating number of G is sharply concentrated around its expected value, in other words a.s. $D(G) = (1 + o(1))\mathbf{Exp}(D(G))$.

For the remainder of the paper, we will use G and F to respectively denote a graph selected from $\Omega_{G,n}$ and the multigraph $H \cup M$ where (H, M) is a pair selected from $\Omega_{F,n}$.

Although we are most interested in results concerning G, we will devote most of our efforts towards analyzing F. As we will show in the next section, if a property a.s. holds for F, then it a.s. holds for G. Our techniques take advantage of the structure of F in a manner which makes it more convenient to work with F than to work directly with G.

The key to this analysis will be the manner in which we select (H, M). We will use a common method in which we *expose* various properties of (H, M), one at a time.

Suppose that we are to choose x from a probability space Ω with domain X and with the uniform distribution. Set $X_1 = X$. For $i \geq 1$, consider any partition $X_i = Y_{i,1} \cup Y_{i,2} \cup \ldots Y_{i,r_i}$, where the $Y_{i,j}$ are disjoint, and select which part x lies in, choosing $Y_{i,j}$ with probability $|Y_{i,j}|/|X_i|$. Set X_{i+1} equal to the selected part, and repeat this process k times, for an arbitrary finite k. Finally, select x uniformly at random from X_{k+1}. (In practice, often $|X_{k+1}| = 1$.) Note that each member of X will be chosen with probability $|X|^{-1}$.

At each step, we say that we are *exposing* a property of x, and we usually describe this process in terms of these properties. For example, when we select (H, M) during the analysis in the proof of Lemma 2, the first property that we expose is H. That is, in the first partition, $Y_{1,j}$ is the set of all pairs with $H = H_j$, for some enumeration of the Hamilton cycles on n vertices: $H_1, H_2, \ldots H_{r_1}$, $r_i = (n-1)!/2$.

1 The Models

In this section, we justify analysing F rather than G. Let P be any graph property.

Lemma 1 *If F a.s. does not have P then G a.s. does not have P.*

Proof:

The probability that F has P is $\zeta(n) = o(1)$.

Consider the probability space $\Omega'_{F,n}$, whose domain is Σ'_n the set of all *edge-disjoint* pairs (H, M) where H is a Hamilton cycle and M is a perfect matching, both with vertex set $[n]$, and which has the uniform distribution. We select a random 3-regular graph F' in a manner analagous to that with which we select F: we choose a member of $\Omega'_{F,n}$ and set $F' = H \cup M$.

It is easily verified that the number of multiple edges in F is asymptotic to a Poisson variable with mean 1. Therefore, the probability that F is simple is $e^{-1} + o(1)$, and thus F' a.s. does not have P, i.e. the probability that F' has P is $\gamma(n) = o(1)$.

Consider the bipartite graph B with vertex set (Υ_n, Σ'_n), and edge set $\{(G, (H, M)) \mid G = H \cup M\}$. Thus, each component of B is a star. We will consider $(H, M) \in \Sigma'_n$ to have P if $H \cup M$ has P.

The results of [7] imply something much stronger than G is a.s. Hamiltonian. They imply that the number of Hamilton cycles in G, which we denote by $H(G)$, is a.s. of the same order as $\mathbf{Exp}(H)$, i.e. for any $\rho(n)$ tending to infinity with n, a.s. $\mathbf{Exp}(H)/\rho(n) < H(G) < \rho(n)\mathbf{Exp}(H)$. Thus, if $\Upsilon^*_n \subset \Upsilon_n$ is the set of 3-regular graphs G for which $\gamma(n)^{1/3}\mathbf{Exp}(H) \leq H(G) \leq \frac{\mathbf{Exp}(H)}{\gamma(n)^{1/3}}$, then

$$\frac{|\Upsilon^*_n|}{|\Upsilon_n|} = 1 - o(1).$$

Now consider the neighbourhood $N(\Upsilon^*_n)$ in B. The probability that a uniformly random element of $N(\Upsilon^*_n)$ has P is at most $\gamma(n)(1 + o(1))$.

Given any two graphs $G_1, G_2 \in \Upsilon^*_n$, $|N(G_1)|/|N(G_2)| \geq \gamma(n)^{2/3}$. Therefore, the probability that a uniformly random member of Υ^*_n has P is $O(\gamma(n)^{1/3})$.

Therefore, the probability that a uniformly random member of Υ_n has P is of order $O(\gamma(n)^{1/3}) = o(1)$, and so G a.s. does not have P. $\square$

We will now show that the dominating number of G is concentrated. By Lemma 1 it is enough to prove:

Lemma 2 *A.s.* $D(F) = (1 + o(1))\mathbf{Exp}(D(F))$.

This of course will imply that a.s. $D(G) = (1 + o(1))\mathbf{Exp}(D(G))$.

To prove Lemma 2, we will make use of the following theorem of Azuma:

Azuma's Inequality [2] *Let* $0 = X_0, \ldots, X_n$ *be a martingale with*

$$| X_{i+1} - X_i | \leq 1$$

for all $0 \leq i < n$. *Let* $\lambda > 0$ *be arbitrary. Then*

$$Pr[| X_n | > \lambda\sqrt{n}] < e^{-\lambda^2/2}$$

This yields the following very useful standard corollary.

Corollary *Let* $Y_1, Y_2, \ldots, Y_n$ *be a sequence of random events. Let* $f = f(Y_1, Y_2, \ldots, Y_n)$ *be a random variable defined by these* Y_i. *If for each* i:

$$\max | \mathbf{Exp}(f \mid Y_1, Y_2, \ldots Y_{i+1}) - \mathbf{Exp}(f \mid Y_1, Y_2, \ldots Y_i) | \leq c_i$$

then the probability that $| f - \mathbf{Exp}(f) | > t$ *is at most:*

$$2 \exp\left(\frac{-t^2}{2\sum c_i^2}\right)$$

For more details on this corollary and an excellent discussion of Martingale arguments see either [5] or [3].

Now,

Proof of Lemma 2:

We shall first expose H, and then expose M, one edge at a time. More specifically, if $V(F) = \{v_1, \ldots, v_n\}$ then we repeatedly expose the partner of the vertex with the lowest index of those vertices whose partners in M are not yet exposed. Note that $\mathbf{Exp}(D \mid H)$ is unaffected by our choice of H.

At any step of this exposure, let M' be the set of edges already known to be in M. Suppose that we are about to expose the partner of v_j, and that v_l, v_m are two potential choices.

Claim:

$$|\mathbf{Exp}(D \mid M' \cup \{(v_j, v_l)\} \subseteq M) - \mathbf{Exp}(D \mid M' \cup \{(v_j, v_m)\} \subseteq M)| \leq 1.$$

Proof of Claim: Let Γ_1 be the set of possibilities for F upon selecting the edge (v_j, v_l), and let Γ_2 be the set of possibilities upon selecting the edge (v_j, v_m). Consider the one-to-one mapping $\phi : \Gamma_1 \to \Gamma_2$ where for each $F_1 \in \Gamma_1$, $F_1, \phi(F_1)$ differ only in that the edges $(v_j, v_l), (v_m, u)$ in F_1 are replaced by $(v_j, v_m), (v_l, u)$ in $\phi(F_1)$.

Clearly $|D(F_1) - D(\phi(F_1))| \leq 1$, proving the claim.

Therefore, by the corollary of Azuma's Inequality,

$$\mathbf{Pr}\{|D(F) - \mathbf{Exp}(D)| \geq n^{2/3}\} \leq 2\ \mathrm{e}^{-n^{\frac{1}{3}}}.$$

$\square$

2 Small Dominating Sets

In this section we describe a simple algorithm which finds a dominating set, S, in a multigraph F which is the union of a Hamilton cycle on n vertices, and a perfect matching on the same vertex set. We then prove that if the Hamilton cycle and matching are selected independently and uniformly at random from amongst all possible choices, then a.s. $|S| \leq .3126n$.

The algorithm implements the straightforward idea discussed in the introduction. Its formal specification is:

DOMINATE:

Input: A Hamilton cycle, $H = (x_1, x_2, \ldots, x_n)$
and a perfect matching M on the same vertex set.
Output: A dominating set S of the multigraph $F = H \cup M$.

begin
  $S := \{x_1\}$;
  $i := 3$;
  while $i < n$ **do**
  begin
A: **if** $(x_i, x_j) \in M$ for some $x_j \in S$ **then** $i := i + 1$
   else
   begin
    $S := S \cup \{x_{i+1}\}$;
    $i := i + 3$;
   end
  end
end

We wish to determine $k_j = |S|$ after j iterations of DOMINATE. In order to do this, we must be very specific about the order in which we expose the edges of M. We also need to introduce another set and some auxiliary bookkeeping variables:

- A - the set of vertices which were examined in line A of DOMINATE and were found not to be adjacent to S via M.
- a - the number of edges of M between A and S.
- e - the number of edges of M between members of S
- s - the number of times that a vertex was examined in line A of DOMINATE and found to be already adjacent to S via M.

The algorithm DOM does this.

DOM:

Input: A random pair (H, M) selected from $\Omega_{F,n}$.
No properties of (H, M) are initially exposed.
Output: A dominating set S of the multigraph $F = H \cup M$.

begin
  Expose H, and relabel the vertices sequentially around H: $x_1, x_2, \ldots x_n$;
  $S := \{x_1\}$;
  $A := \emptyset$;
  $i := 3$;
  $a := 0$;
  $e := 0$;
  $s := 0$;
  while $i < n$ **do**
  begin
A: Ask "Is x_i adjacent to some member of S via M?"
    if "YES" **then**
    begin
      Expose the partner of x_i;

```
        i := i + 1;
        s := s + 1;
      end
      else
      begin
        S := S ∪ {x_{i+1}};
        A := A ∪ {x_i};
B:      Ask "Is x_{i+1} adjacent to some member of S via M?"
        if "YES" then
        begin
          Expose the partner of x_{i+1};
          e := e + 1;
        end
        else
        begin
C:        Ask "Is x_{i+1} adjacent to some member of A via M?"
          if "YES" then
          begin
            Expose the partner of x_{i+1};
            a := a + 1;
          end
        end
        i := i + 3;
      end
    end
end
```

It is important to note that whenever one of the questions A, B, or C is asked, the partner of the vertex in question is *not* exposed. the only thing exposed is the answer, "Yes" or "No".

Now the key to analyzing DOM, is determining the probabilities of the questions asked being answered "Yes". These probabilities will clearly depend on the stage of the algorithm and on the values of the various variables.

Let i_j, a_j, e_j, s_j be the values of i, a, e, s at the beginning of the jth iteration of the **while** loop, and let $k_j = |A|+1 = |S|$ at the same point. Note that $j = s_j + k_j$, and $i_j = s_j + 3k_j$. Let p_j, r_j, q_j be the probabilities that questions A, B, and C, respectively are answered "Yes" if asked during the jth iteration.

Now, at this point, we have exposed only the following information about M:

- The set of edges of M which lie between A and S. Its size is a_j.
- The set of edges of M which lie entirely within S. Its size is s_j.
- The partners in M of exactly s_j other members of S.

Upon substituting $s_j = j - k_j$, we claim:

$$\begin{aligned}
p_j = p(j, k_j, a_j, e_j) &= \frac{k_j - a_j - s_j - 2e_j}{n - 2k_j - s_j} \\
&= \frac{2k_j - a_j - 2e_j - j}{n - k_j - j} \\
r_j = r(j, k_j, a_j, e_j) &= \frac{2k_j - a_j - 2e_j - j}{n - k_j - j - 1} \\
q_j = q(j, k_j, a_j, e_j) &= \frac{k_j - a_j + 1}{n - 2k_j + 2e_j - 1}
\end{aligned}$$

Proof of Claim:

For p_j: there are $k_j - a_j - s_j - 2e_j$ current members of S whose partners in M are presently unexposed. There are $n - 2k_j - s_j$ vertices including x_j which are eligible to be partners of vertices in S, each of which is equally likely. A similar argument explains r_j. For q_j: if we exposed the partners of S, then we would be left with $n-2k_j+2e_j-1$ vertices which are eligible to be the partner of x_{j+1} in M, each of which is equally likely. $k-a_j+1$ of them are in A. □

Consider the conditional expected change of these variables at each iteration:

$$\begin{aligned}
\mathbf{E}(k_{j+1}-k_j) &= 1-p_j,\\
\mathbf{E}(a_{j+1}-a_j) &= (1-p_j)(1-r_j)q_j,\\
\mathbf{E}(e_{j+1}-e_j) &= (1-p_j)r_j.
\end{aligned}$$

From these we obtain the following system of differential equations:

$$\begin{aligned}
K(0) &= A(0)=E(0)=0,\\
K^{'}(x) &= 1-\frac{2K(x)-A(x)-2E(x)-x}{1-K(x)-x},\\
A^{'}(x) &= (1-\frac{2K(x)-A(x)-2E(x)-x}{1-K(x)-x})^2(\frac{K(x)-A(x)}{1-2K(x)+2E(x)}),\\
E^{'}(x) &= (1-\frac{2K(x)-A(x)-2E(x)-x}{1-K(x)-x})(\frac{2K(x)-A(x)-2E(x)-x}{1-K(x)-x}).
\end{aligned}$$

It will follow from a theorem of Wormald [9] that for $j=(x+o(1))n$, a.s. $k_j=(K(x)+o(1))n, a_j=(A(x)+o(1))n$, and $e_j=(E(x)+o(1))n$. Using this, we can determine the value x^* such that we a.s. take $(x^*+o(1))n$ steps to go completely around H. Thus, it follows that a.s. S is of size $(K(x^*)+o(1))n$.

Before stating that theorem, we will briefly discuss why this should be true.

Suppose n is fixed and large. Plot the Cartesian points $P_j=(j/N,k_j/N)$. Now the slope of the line segment (P_j,P_{j+1}) is precisely $k_{j+1}-k_j$, and hence it has expected value $1-p_j$.

This is why we choose $K(x)$ so that $K^{'}(x)$ is $1 - p(xn, K(x)n, A(x)n, E(x)n)$. $A(x)$ and $E(x)$ are chosen for similar reasons.

Now to show that the points P_j a.s. lie very close to the curve $K(x)$, it is not enough to show that the slope is chosen correctly, we must also show that k_j is a.s. very close to $\mathbf{Exp}(k_j)$.

Recall that Azuma's inequality essentially shows that if a random variable X is selected via a (long) sequence of random events, and if each event can only affect X by a small amount then with high probability X is very close to $\mathbf{Exp}(X)$. Each time we ask a question during DOM, the answer can only affect each of the variables by at most one. Thus (at least intuitively) with high probability they won't differ from their expected values by very much throughout the entire run of DOM.

We will use here a slightly weaker version of the theorem as it appears in [9].

Let $Y_j^{(i)}$ be a sequence of real-valued random variables, $i = 0 \dots b$, for some fixed b, such that $0 \leq Y_j^{(i)} \leq Cn$ for some constant C. Let H_j be the history of the sequence, i.e. the array $(Y_0, \dots, Y_j)$, where Y_j is the vector $(Y_j^{(1)}, \dots, Y_j^{(b)})$.

In the following theorem, statements concerning $Y_j^{(i)}$ are only applicable for $j \leq m$. Note that "uniformly" refers to the convergence implicit in the $o()$ terms. Hypothesis (i) ensures that $Y^{(i)}$ does not change too quickly throughout the process, (ii) tells us what we expect the rate of change to be, and (iii) ensures that this rate does not change too quickly.

Theorem 1 *Suppose that $m = m(n) \leq Bn$ for some constant B. For $1 \leq l \leq b$, let $f_l : \mathbb{R}^{a+1} \rightarrow \mathbb{R}$. Suppose also that*

(i) there exists constant d such that for all l, uniformly for all t

$$\mathbf{Pr}\left(|Y_{t+1}^{(l)} - Y_t^{(l)}| > d |\, H_t\right) = o(n^{-3})$$

always;

(ii) for all l and uniformly over all t,

$$\mathbf{Exp}(Y_{t+1}^{(l)} - Y_t^{(l)} | H_t) = f_l(t/n, Y_t^{(1)}/n, \ldots, Y_t^{(a)}/n) + o(1)$$

always;

(iii) for each l the function f_l is continuous and satisfies a Lipschitz condition on D, where D is some bounded connected open set containing

some ϵ-neighbourhood of $\{(0, y^{(1)}, \ldots, y^{(a)}) : \mathbf{Pr}(Y_0^{(l)} = y^{(l)}, 1 \le l \le a) \ne 0\}$.

Then

(a) for $(0, \hat{y}^{(1)}, \ldots, \hat{y}^{(a)}) \in D$ the system of differential equations

$$\frac{dz_l}{ds} = f_l(s, z_1, \ldots, z_a), \qquad i = 1, \ldots, a$$

has a unique solution in D for $z_l : \mathbb{R} \to \mathbb{R}$ passing through

$$z_l(0) = \hat{y}^{(l)}/n,$$

$1 \le l \le a$,

and which extends to points arbitrarily close to the boundary of D;

(b) almost surely

$$Y_t^{(l)} = Z_l(t/n)n + o(n)$$

uniformly for $0 \le t \le \min\{\sigma n, m\}$ and for each l, where where $Z_l(t)$ is the solution in (a) with $\hat{y}^{(l)} = Y_0^{(l)}$, and $\sigma = \sigma(n)$ is the supremum of those s to which the solution can be extended.

Recall that a function $f(u_1, \ldots, u_j)$ satisfies a *Lipschitz condition* on $D \subseteq \mathbb{R}^j$ if a constant $L > 0$ exists with the property that

$$|f(u_1,\dots,u_j)-f(v_1,\dots,v_j)|\le L\sum_{i=1}^{j}|u_i-v_i|$$

for all $(u_1,\dots,u_j)$ and $(v_1,\dots,v_j)$ in D.

We now apply this theorem, setting $Y_j=(k_j,a_j,e_j)$ and f_1,f_2,f_3 congruent to $K^{'},A^{'},E^{'}$ respectively. The conditions hold by taking $B=C=d=1$, $\hat{y}^{(1)}=\hat{y}^{(2)}=\hat{y}^{(3)}=0$, and $D=\{(x_1,x_2,x_3,x_4)\mid -\epsilon<x_i<1,-\epsilon<x_1+2x_2<1-\epsilon\}$, for any small $\epsilon>0$.

After j steps, we have passed $s_j+3k_j=j+2k_j$ vertices on H. Thus, we wish to solve for x such that $x+2K(x)=1$, implying that a.s. the size of S is approximatly $K(x)n$. However, we can't quite do this, as in order to ensure that a Lipschitz condition holds on D, we can only extend D to include values of x such that $x+2K(x)\le 1-\epsilon$ for some $\epsilon>0$. Thus, we choose some x such that $x+2K(x)=1-\epsilon$, and consider the dominating set that we get by halting DOMINATE after reaching $x_{(1-\epsilon)n}$ and then adding the remaining ϵn vertices to S, which will a.s. have size $\approx (K(x)+\epsilon)n$.

Using Maple, we find that for $x=.3807, K(x)=.3125\ldots$, and $x+2K(x)=.9999466\ldots$, which implies:

Theorem 2 *If G is a random 3-regular graph on n vertices, then $D(G)<.3126n$.*

We can use DOMINATE to obtain a randomized algorithm, which when presented with a random 3-regular graph G, will a.s. output a dominating set of size at most $.3126|V(G)|$.

In [4] we have a randomized algorithm which takes as input a random k-regular graph G ($k\ge 3$), and a.s. outputs a Hamilton cycle chosen nearly uniformly at random from all Hamilton cycles of G. If we use this algorithm to choose a pair $(\bar{H},\bar{M})$ whose union is G such that no such pair is more than twice as likely to be chosen as any other pair, then it follows as in the proof of Lemma 1, that $(\bar{H},\bar{M})$ is pseudoequivalent to a uniformly random edge-disjoint pair (H,M). Therefore, if we apply DOMINATE to $(\bar{H},\bar{M})$, we will a.s. find such a dominating set.

3 Lower Bound

In this section we will show:

Theorem 3 *If G is a random 3-regular graph on n vertices, then G a.s. has no dominating set of size less than $.2636n$.*

Proof:

Again, we will work with F. Consider a random $F = (H, M)$ on n vertices. Let $D(F, k)$ be the number of dominating sets of size k in F. We will compute $\mathbf{Exp}(D(F, k))$, where $k = \lceil \kappa n \rceil$ for some fixed $\frac{1}{4} < \kappa < \frac{1}{3}$. We will then see that when $\kappa = .2636$, $\mathbf{Exp}(D(F, k)) = o(1)$, thus proving the theorem.

Consider a potential dominating set X, $|X| = k$.

Expose H. Let a be the number of pairs of vertices in X which are adjacent along H, and let b be number of pairs which are endpoints of a subpath of length 2 in H, whose interior vertex is not in X. We will determine $P_n(a, b, k)$, the probability that X is a dominating set.

Let S_X be the set of vertices which are not adjacent to X in H. The cardinality of S_X is exactly $n + 2a + b - 3k$, and if X is a dominating set then $|S_X| \leq k$. Thus if $P_n(a, b, k) > 0$, then a, b must satisfy $2a + b \leq 4k - n$, and so we let $Q_k = \{(a, b) \mid a, b \geq 0, 2a + b \leq 4k - n\}$.

Now $P_n(a, b, k)$ is simply the probability that every vertex in S_X is adjacent to a vertex in X via M. Therefore:

$$P_n(a, b, k) = \binom{k}{n + 2a + b - 3k} (n + 2a + b - 3k)! \frac{M(6k - 4a - 2b - n)}{M(n)}$$

where for even l, $M(l) = l!/2^{l/2}(l/2)!$ is the number of ways to pair up l points.

Now, the number of choices for X for a given choice of k and $(a, b) \in Q_k$ is clearly independent of our choice of H, and we claim that it is equal to:

$$C_n(k,a,b) = \frac{n(k-a-b)}{k(n+a-2k)} \binom{k}{a, b, k-a-b} \binom{n+a-2k}{k-a-b}$$

Proof of Claim:

Choose any initial vertex x_1, and fix an orientation of H, inducing a directed path P. Let $\mathcal{X}$ be the set of choices for $X \subseteq V(P)$ containing x_1. We show that $|\mathcal{X}|$ is $C_n(k,a,b)k/n$; the result follows by symmetry.

Consider the set

$$\begin{aligned} \mathcal{L} = \{(\ell_1, \ell_2, \ldots, \ell_k) \quad | \quad & \sum_{i=1}^{k} \ell_i = n \\ & \ell_i \geq 1, \\ & \ell_i = 1 \quad \text{for exactly } a \text{ values of } i, \\ & \ell_i = 2 \quad \text{for exactly } b \text{ values of } i\} \end{aligned}$$

Clearly, there is a bijection from $\mathcal{L}$ onto $\mathcal{X}$, and it is easily verified that

$$\begin{aligned} |\mathcal{L}| &= \binom{k}{a, b, k-a-b} \binom{n+a-2k-1}{k-a-b-1} \\ &= \frac{k-a-b}{n+a-2k} \binom{k}{a, b, k-a-b} \binom{n+a-2k}{k-a-b} \end{aligned}$$

which proves the claim. $\square$

Therefore,

$$\begin{aligned} \mathbf{Exp}(D(F,k)) &= \sum_{a,b \in Q_k} \frac{n(k-a-b)}{k(n+a-2k)} \binom{k}{a, b, k-a-b} \binom{n+a-2k}{k-a-b} \\ & \quad \times \binom{k}{n+2a+b-3k} (n+2a+b-3k)! \frac{M(6k-4a-2b-n)}{M(n)} \end{aligned}$$

$$\leq \sum_{a,b \in Q_k} O(1) \times T(a,b)$$

where

$$T(a,b) = \frac{(k!)^2}{M(n)a!b!(k-a-b)!^2} \times \frac{(n+a-2k)!(6k-4a-2b-n)!}{(n+2a+b-3k)!(4k-n-2a-b)!(3k-2a-b-\frac{n}{2})!2^{3k-2a-b-\frac{n}{2}}}$$

Now we will show that for $\kappa \leq .2636$, $\mathbf{Exp}(D(F,k)) = o(1)$, which implies that a.s. $D(F,k) = 0$. We will do this by showing that for any such κ there is some $\gamma < 1$ so that $T(a,b) < \gamma^n$ for $(a,b) \in Q$.

We will approximate $T(a,b)$ using Stirling's Formula. We will first show that for each term $\omega!$ in $T(a,b)$, we can ignore the case where ω is of order $o(n)$.

Note that for $(a,b) \in Q_k$, $n+a-2k \geq n/3$, $6k-4a-2b-n \geq n/3$, and for $\delta = 1-3\kappa > 0$. $k-a-b \geq \delta n$, and $n+2a+b-3k \geq \delta n$.

Now we will show that for some $\epsilon > 0$, $T(a,b)$ attains its maximum on $Q_k^{'} = \{(a,b) \mid a,b \geq \epsilon n, 2a+b \leq 4k-n-\epsilon n\}$. Note that for small enough ϵ, $\{(a,b) \mid a,b,4k-n-(2a+b) \leq \epsilon n\} = \emptyset$.

First, suppose $4k-n-(2a+b) > \epsilon n$.

$$\begin{aligned}
\frac{T(a,b)}{T(a+1,b)} &\sim \frac{2(6k-4a-2b-n)^2(a+1)(n+2a+b-3k)^2}{(n+a-2k)(k-a-b)^2(4k-n-2a-b)^2} \\
&= a\theta(n^{-1}) \\
&< 1
\end{aligned}$$

for $a < \epsilon n$.

Similarly, $T(a,b)/T(a,b+1) < 1$ for $b < \epsilon n$.

To deal with the case $4k-n-(2a+b) < \epsilon n$, we make the substitution $t = 2a+b$. Now, suppose $b = t-2a > \epsilon n$.

$$
\begin{aligned}
\frac{T(a,t)}{T(a+1,t)} &\sim \frac{(a+1)(k+a-t)^2}{(n+a-2k)(t-2a)^2} \\
&< 1
\end{aligned}
$$

for $a < \epsilon n$.

$$
\begin{aligned}
\frac{T(a,t+1)}{T(a,t)} &\sim \frac{(k+a-t)^2(4k-n-t)}{(6k-2t-n)(t-2a)(n+t-3k)} \\
&< 1
\end{aligned}
$$

for $t < \epsilon n$.

The case $4k-n-(2a+b) < \epsilon n, a > \epsilon n$ is dealt with in a similar manner, thus establishing that $T(a,b)$ attains its maximum on $Q_k^{'}$.

Now, let $\alpha = a/n, \tau = (2a+b)/n$. It follows from Stirling's formula that:

$$T(a,b) \sim \text{poly}(n) F(\alpha,\tau)^n$$

for $(a,b) \in Q_k$, where

$$F(\alpha,\tau) = \frac{\kappa^{2\kappa}(1-2\kappa+\alpha)^{1-2\kappa+\alpha}(6\kappa-1-2\tau)^{3\kappa-1/2-\tau}}{\alpha^{\alpha}(\tau-2\alpha)^{\tau-2\alpha}(\kappa+\alpha-\tau)^{2\kappa+2\alpha-2\tau}(1-3\kappa+\tau)^{1-3\kappa+\tau}(4\kappa-1-\tau)^{4\kappa-1-\tau}}$$

At any maximal point of F, the derivatives of $\ln F$ with respect to α and τ are both zero, yielding:

$$
\begin{aligned}
(1-2\kappa+\alpha)(\tau-2\alpha)^2 - \alpha(\kappa+\alpha-\tau)^2 &= 0 \\
(\kappa+\alpha-\tau)^2(4\kappa-1-\tau) - (6\kappa-1-2\tau)(\tau-2\alpha)(1-3\kappa+\tau) &= 0
\end{aligned}
$$

When we take $\kappa = .2636$ we find that there is only one solution corresponding to $(a,b) \in Q_k$: $\alpha = \alpha^* = .002115\ldots$, $\tau = \tau^* \sim .020589\ldots$. This is a local maximum, and $F(\alpha^*,\tau^*) = .991914\ldots$, and the result follows. $\Box$

Note: This result can be improved slightly by computing the expected number of *minimal* dominating sets. The improvement is quite small (in fact, it doesn't change the first four digits), and the analysis is somewhat involved, so we don't include it here.

References

[1] N. Alon. Private communication.

[2] K. Azuma. *Weighted Sums of Certain Dependent Random Variables.* Tokuku Math. Journal **19** (1967), 357 - 367.

[3] B. Bollobás, *Martingales, Isoperimetric Inequalities and Random Graphs.* Colloq. Math. Soc. Janós Bolyai 52 (1987), 113 - 139.

[4] A. Frieze, M. Jerrum, M. Molloy, R. W. Robinson and N. C. Wormald. *Generating and Counting Hamilton Cycles in Random Regular Graphs.* Manuscript.

[5] C. McDiarmid. *On the Method of Bounded Differences.* Surveys in Combinatorics, Proceedings of the Twelfth British Combinatorial Conference (1989), 148 - 188.

[6] B. Reed. *Paths, Stars, and the Number Three.* Manuscript.

[7] R. W. Robinson and N. C. Wormald. *Almost All Cubic Graphs are Hamiltonian.* Random Structures and Algorithms **3** (1992), 117 - 126.

[8] R. W. Robinson and N. C. Wormald. *Almost All Regular Graphs are Hamiltonian.* Random Structures and Algorithms (to appear).

[9] N. Wormald. *Differential Equations for Random Processes and Random Graphs.* Manuscript.