CSC C63  Winter 2018, Assignment #4

Due: Wed April 4 at 11:00 PM. You can submit it up to two days late with a penalty of 2% per day. You must submit a .pdf file to MarkUs.

You may work with a partner to solve the problems. You must write your own solution. You may not look at your partner's written solution. You may not consult with your partner on how to write your solution. You can only have one partner. Neither you nor your partner can discuss this assignment with anyone else other than the course instructor and the TA. If you work with a partner then you must name your partner on your assignment.

This assignment has 2 pages

You can assume as fact that the following problems are NP-complete: SAT, 3-SAT, IND-SET, CLIQUE, VERTEX-COVER, SUBSET-SUM, HAMPATH (both the directed and undirected versions).

In some problems, you are asked to give a good reason for your answer. Your mark will be based on the strength of the reason. A proof is a very strong reason, but is not always possible. Proving something like "otherwise, P=NP" is not quite as strong as a proof, but is still a very strong reason. Saying "I think it's NP-complete because it looks a bit like CLIQUE" is a weak reason.

The graphs in this assignment are all undirected.

1. (30 pts) For each of the following, state whether you think that the problem is NP-complete. Justify your answer by either proving that it is NP-complete, or giving a good reason to think that it is not.

   (a) **LONG-PATH**
      
      **Input:** A graph $G$ with two specified vertices $u, v$, and a target $T$.
      
      **Question:** Does $G$ have a path from $u$ to $v$ of length at least $T$?
      
      **Clarification:** A path does not repeat any vertices. Its length is the number of edges.

   (b) **10-VERTEX-COVER**
      
      **Input:** A graph $G$.
      
      **Question:** Does $G$ have a vertex cover of size at most 10?

   (c) **CLIQUE-AND-IND-SET**
      
      **Input:** A graph $G$ and a target $T$.
      
      **Question:** Does $G$ have a clique and an independent set whose sizes sum to at least $T$?
      
      **Clarification:** The clique and independent set do not have to be disjoint. If a vertex is in both, then it contributes two to the sum of the sizes.

2. (20 pts) Exercise 7.29 of Sipser (3rd edition). This is Problem 7.27 of Sipser (2nd edition). Here, you will show that 3COLOR is NP-complete by reducing it from 3-SAT. You can skip the step where you show that 3COLOR $\in$ NP.

   **Comment:** Moore-Mertens provides a different proof that 3COLOUR is NP-complete by reducing it from NAE-SAT. It may be helpful to read that, but for your assignment you need to give a reduction from 3-SAT using the subgraphs provided in the Sipser exercise.

3. (15 pts)

   A subset $A$ of the vertices of a graph is a maximal clique if (i) $A$ is a clique and (ii) $A$ is not a subset of a bigger clique.

   Show that the following problem can be solved in PSPACE:

   **MANY-MAXIMAL-CLIQUE**
   
   **Input:** A graph $G$ and an integer $T$.
   
   **Question:** Does $G$ contain at least $T$ maximal cliques?
4. **(25 pts)** In this problem, you need to verify the answer to a multiplication.

**MULTIPLICATION**

**Input:** Three integers $a, b, c$ presented in binary

**Question:** Is $a \times b = c$?

Prove that MULTIPLICATION is in $L$.

*Clarification:* $a, b, c$ are written in binary on the input tape. They are separated by a hash-character. Note that the size of the input is the total number of digits in $a, b, c$. So for your algorithm to run in $L$, the space used must be a constant multiple of the logarithm of the total number of digits.

5. **(20 pts)** A walk from $s$ to $t$ of length $\ell$ in a graph is a sequence of vertices $v_0, \ldots, v_\ell$ where $v_0 = s, v_\ell = t$ and $v_i, v_{i+1}$ is joined by an edge for every $0 \leq i \leq \ell - 1$. So the only difference between a walk and a path is that a walk may repeat vertices.

**WALKS-OF-MANY-LENGTHS**

**Input:** An undirected graph $G$ with two specified vertices $s, t,$ and two integers $1 \leq L_1 \leq L_2 \leq n$, where $n$ is the number of vertices in $G$.

**Question:** Does $G$ have a walk from $s$ to $t$ of length $\ell$ for every $L_1 \leq \ell \leq L_2$?

So, for example, if $L_1 = 6, L_2 = 9$ then we are asking whether there are four walks of lengths 6, 7, 8 and 9.

Prove that WALKS-OF-MANY-LENGTHS $\in \text{NL}$ by providing a logspace NDTM that accepts WALKS-OF-MANY-LENGTHS.

*Note:* Since $L_1 \leq L_2 \leq n$, we know that the input file has size at least $L_2$. So in this problem the NDTM is allowed to use $O(\log L_2)$ bits of space. (Do you understand why?)