**CSC C63** Assignment #3 2018

**Due:** Fri, Mar 16 at 11:00 PM. **You must submit a .pdf file** to MarkUs.

You may work with a partner. **You must write your own solution. You may not look at your partner's written solution.** You can only have one partner. Neither you nor your partner can discuss this assignment with anyone else other than the course instructor and the TA. If you work with a partner then you must name your partner on your assignment.

1. **(30 pts)** In this problem, you have a machine that can run for a fixed amount of time. You have several job requests, each request specifies the amount of time the job takes and the payment for it. You want to choose which jobs to take so that (i) your machine can run them all and (ii) the total payment is as high as possible.

   **JOBS**
   Input: For each job $i$, you are given its time $t_i$ and payment $p_i$. You are also given the amount of time that your machine can run, $T$. These numbers are all non-negative integers.
   Output: An optimal set of jobs. I.e. a set $J$ such that $\sum_{i \in J} t_i \leq T$ and $\sum_{i \in J} p_i$ is as high as possible.

   Find a Yes/No problem in NP which is equivalent to JOBS. (Meaning that you can prove (i) and (ii) below.)

   Prove (i) that any polytime algorithm for your problem can be used to provide a polytime algorithm for JOBS. and (ii) that any polytime algorithm for JOBS can be used to provide a polytime algorithm for your problem.

   Then prove that your problem is in NP, using a nondeterministic Turing machine.

2. **(8 pts)** Prove that the following problem is in co-NP, using either a verification algorithm or a non-deterministic Turing machine:

   **PRIME**
   Input: A positive integer $x \geq 2$.
   Question: Is $x$ prime?

3. **(15 pts)**

   **HALF-HAM**
   Input: A graph $G$ with specified vertices $u, v$.
   Question: Is there a path in $G$ from $u$ to $v$ that uses exactly half of the vertices in $G$?

   Recall that a path cannot repeat any vertices.

   For example, if $G$ has 20 vertices, including $u, v$, then the path would need to consist of $u, v$ and 8 other vertices. Note that if $G$ has an odd number of vertices, then the answer is always NO. So the only interesting inputs will have an even number of vertices.

   (a) Prove that HALF-HAM is in NP using a verification algorithm.
   (b) Prove that HAM-PATH $\leq_P$ HALF-HAM.

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4. **(25 pts)** The Partition Problem is defined as follows:

   **PARTITION**
   Input: A list of integers $y_1, ..., y_n \geq 0$.
   Question: Can you partition $\{1, .., n\}$ into two disjoint sets $A_1, A_2$ that have the same sum; i.e. so that $\sum_{i \in A_1} y_i = \sum_{i \in A_2} y_i$?

   (a) Prove that PARTITION is in NP using a verification algorithm.

   (b) Prove that PARTITION is in NP using a nondeterministic Turing machine.

   (c) Prove that SUBSET-SUM $\leq_P$ PARTITION. This is Problem 4.13 of Moore and Mertens, who give the hint "add an element to the list".

   SUBSET-SUM is defined on page 269 of the 2nd Ed of Sipser, page 296 of the 3rd Ed, and on 105 of Moore and Mertens. You should assume that the input to SUBSET-SUM are all non-negative integers (this is not clear in Sipser). Note that Moore and Mertens say "INTEGER PARTITIONING" rather than "PARTITION".

5. **(10 pts)** Prove that for every language $L \in NP$, there is a constant $c$ such that $L$ can be decided in time $O(2^{n^c})$.

6. **(10 pts)** Prove that if $P = NP$ then $P = co - NP$.