You may work with a partner. You must write your own solution. You may not look at your partner’s written solution. You can only have one partner. Neither you nor your partner can discuss this assignment with anyone else other than the course instructor and the TA. If you work with a partner then you must name your partner on your assignment.

For problems 1 - 6, answer: (i) is the set decidable? (ii) is the set recognizable? You must prove your answers.

The variables $P, Q$ represent algorithms (formally: Turing Machines with outputs) which have positive integers as input and output (formally: the input and output are binary strings which are interpreted as integers). The variable $R$ represents an algorithm that has a pair of positive integers as input and a single positive integer as output. The variables $x, y, k, \ell$ represent positive integers.

(20 marks each)

1. $\langle P \rangle, \langle Q \rangle, k$ there are at least $k$ natural numbers $x$ for which $P(x), Q(x)$ both halt.
2. $\langle P \rangle, k$ there are at most $k$ natural numbers $x$ for which $P(x)$ outputs 7.
3. $\langle R \rangle$ for every input $(x, y)$, $R(x, y)$ outputs $x + y + 25$.
4. $\{x, k\}$ there are at least $k$ Mersenne primes less than $2x$.
5. $\langle P \rangle, \ell$ there is some $x > y \geq \ell$ such that $P(x), P(y)$ both halt and their outputs sum to $x - y + 10$.
6. $\langle P \rangle, \langle Q \rangle, x|P(x)$ runs for at least as many steps as $Q(x)\rangle$.

(If $P(x)$ loops forever, then we consider it to run for $\infty$ steps. So if $P(x), Q(x)$ both loop forever then $(P, Q, x)$ belong to the set. But if $P(x)$ halts and $Q(x)$ loops forever, then $(P, Q, x)$ do not belong to the set.)

7. (25 marks) When $A, B$ are languages, $(AB)^*$ is defined to be the set of strings of the form $a_1b_1a_2b_2\ldots a_kb_k$ where each $a_i$ is in $A$ and each $b_i$ is in $B$. For example, if $A = \{xxx, zyy\}, B = \{yx, xzz\}$ then $zyyyxxxxzzz \in (AB)^*$ as it can be split into the strings $zyy, yx, xxx, zzz$, but $zyyx \notin (AB)^*$ and $xxxyxxxxzzz \notin (AB)^*$.

(a) Suppose that $A, B$ are decidable languages. Prove that $(AB)^*$ is decidable.

(b) Suppose that $A, B$ are recognizable languages. Prove that $(AB)^*$ is recognizable.