For problems 1 - 6, (i) is the set decidable? (ii) is the set recognizable? You must prove your answers.

The variables $P, Q$ represent algorithms (formally: Turing Machines with outputs) which have positive integers as input and output (formally: the input and output are binary strings which are interpreted as integers). The variable $R$ represents an algorithm that has a pair of positive integers as input and a single positive integer as output. The variables $x, y, k, \ell$ represent positive integers.

(20 marks each)

1. $\{< P >, k | \text{there are at least } k \text{ natural numbers } x \text{ for which:} \]
   $\quad P(x) \text{ outputs an integer greater than } x^3 \}$.

2. $\{x, k | \text{there are at least } k \text{ pairs of twin primes between } x \text{ and } k^x \}$.

3. $\{< P >, < Q >, k | \text{there are fewer than } k \text{ natural numbers } x \text{ for which:} \]
   $\quad P(x), Q(x) \text{ both halt} \}$.

4. $\{< P >, \ell | \text{there is no } x > y > z \geq \ell \text{ such that:} \]
   $\quad P(x), P(y), P(z) \text{ all halt and their outputs sum to } xyz \}$.

5. $\{< R > | \text{for every input } (x, y), R(x, y) \text{ outputs } x + y + 1 \}$.

6. $\{< P >, < Q >, x | P(x) \text{ runs for at least as many steps as } Q(x) \}$.

   (If $P(x)$ loops forever, then we consider it to run for $\infty$ steps. So if $P(x), Q(x)$ both loop forever then $(P, Q, x)$ belong to the set. But if $P(x)$ halts and $Q(x)$ loops forever, then $(P, Q, x)$ do not belong to the set.)

7. (15 marks) Consider the Turing Machine presented in Example 3.9 of Sipser. The input is 01#011, and the initial state is $q_1$. So the configuration representing the situation before the first step is $q_101#011$.

   Give the configurations representing each of the next 5 steps.