Problem A: (30 pts) Construct a Turing machine that recognizes the language:

\[ A = \{ xy | x \in \{0\}^*, y \in \{1\}^*; |x| > |y| \}. \]

In other words, the members of \( A \) consist of a string of 0’s followed by a string of 1’s where there are more 0’s than 1’s. (It is possible that one of the strings has length zero, so eg. 000 \( \in A \) but 11 \( \notin A \).)

You must describe the machine as formally as the one described in Example 3.9 on page 145 in the text.

Problem B: (15 pts) The Twin Primes Conjecture states that there is an infinite number of pairs of primes of the form \((p, p + 2)\) (eg. \((3,5), (11,13), (281,283)\)). Such a pair is referred to as “twin primes”. Despite being a very well-studied conjecture, nobody knows (yet) whether it is true.

Are the following sets decidable? Prove your answer.

\[ X = \{ i \in \mathbb{Z}^+ | i \text{ is one of a pair of twin primes} \} \]
\[ Y = \{ j \in \mathbb{Z}^+ | \text{there is no pair of twin primes that are both greater than } j \} \]

Note: In 2013, Yitang Zhang proved that there are an infinite number of pairs of primes which differ by at most \( X \), where \( X \) is roughly 70,000,000. This was considered a tremendous breakthrough, as it was the first such result for any constant \( X \). The value of \( X \) has since been lowered to 246.

Problem C: (25 pts) Consider the following subset of all pairs \((< P >, X)\) where \( P \) is an algorithm (i.e. Turing Machine) that can take arbitrary strings as input, \(< P >\) is a description of \( P \), and \( X \) is an input string. \( P(X) \) denotes \( P \) run with input \( X \).

\[ S = \{ (< P >, X) | P(X) \text{ outputs the first character in } X \} \]

(a) Is \( S \) recognizable? Prove your answer.
(b) Is \( S \) decidable? Prove your answer.

Hint: Adapt the proof that was presented in the second lecture.

Problem D: (5 pts) Write Church’s Thesis in your own words.

\( A^* B^* \) is the set of strings of the form \( a_1a_2...a_t b_1b_2...b_\ell \) where each \( a_i \) is in \( A \), each \( b_j \) is in \( B \) and \( t, \ell \geq 0 \).

Problem E: (10 pts) Suppose that \( A, B \) are decidable languages. Prove that \( A^* B^* \) is decidable.

Problem F: (10 pts) Suppose that \( A, B \) are recognizable languages. Prove that \( A^* B^* \) is recognizable.