Edge-Disjoint Cycles in Regular Directed Graphs

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Abstract

We prove that any k-regular directed graph with no parallel edges contains a collection of at least $\Omega(k^2)$ edge-disjoint cycles, conjecture that in fact any such graph contains a collection of at least $\binom{k+1}{2}$ disjoint cycles, and note that this holds for $k \leq 3$.

In this paper we consider the maximum size of a collection of edge-disjoint cycles in a directed graph. We pose the following conjecture:

Conjecture 1: *If G is a k-regular directed graph with no parallel edges, then G contains a collection of $\binom{k+1}{2}$ edge-disjoint cycles.*

We prove two weaker results:

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Theorem 1: *If G is a k-regular directed graph with no parallel edges, then G contains a collection of at least $5k/2-2$ edge-disjoint cycles.*

Theorem 2: *If G is a k-regular directed graph with no parallel edges, then G contains a collection of at least ϵk^2 edge-disjoint cycles, where $\epsilon = \frac{3}{2^{19}}$.*

Note that Theorem 1 implies that Conjecture 1 is true for $k \leq 3$. The proof of Theorem 2 is probabilistic, and we make no attempt to compute the best value of ϵ, as it is clear that our methods will not yield ϵ near $\frac{1}{2}$.

Before proving Theorems 1 and 2, we note that the bound in Conjecture 1, if true, is tight. To see this, consider the directed graph $C_n^k, n \geq 2k+1$, which has vertex set $\{0, \ldots, n-1\}$, and edge set $\{(i, i+j) : 0 \leq i \leq n-1, 1 \leq j \leq k\}$, where the addition is taken mod n. It is easy to see that any cycle in C_k^n must contain one of the $\binom{k+1}{2}$ edges from $\{n-k, \ldots, n-1\}$ to $\{0, \ldots, k-1\}$. (Note that C_n^k does in fact contain $\binom{k+1}{2}$ edge-disjoint cycles, as there are k edge-disjoint cycles through $n-1$, whose deletion, along with the deletion of $n-1$, yields a graph isomorphic to C_{n-1}^{k-1}.) Also, the complete directed graph on $k+1$ vertices has $2\binom{k+1}{2}$ edges, and hence provides another example.

As usual, we use $\delta^+(v)$ and $\delta^-(v)$ to denote the outdegree and indegree repectively, of a vertex v. The maximum (resp. minimum) degree of a graph is the maximum (resp. minimum) of the indegrees and outdegrees of its vertices. We say a graph G is *Eulerian* if $\delta^-(v) = \delta^+(v)$ for all $v \in V(G)$. The degree of a vertex v in an Eulerian graph is $\delta^-(v)$, and an Eulerian graph is *k-regular* if each vertex degree is k. Note that here we do not require an Eulerian graph to be connected.

1 Related Work

Conjecture 1 is reminiscent of a number of other conjectures and theorems.

Behzad, Chartrand and Wall [3] have conjectured that every k-regular digraph on n vertices has a cycle of length at most $\lceil \frac{n}{k} \rceil$. Caccetta and Häggkvist [4] made the stronger conjecture that this is true for every digraph with minimum outdegree k. The most important results in this direction are due to Chvátal and Szemerédi [5] who showed that every digraph with minimum outdegree k on n vertices has a cycle of length at most $\min(\frac{2n}{k+1}, \lceil \frac{n}{k} \rceil + 2500)$. Nishimur [10] refined their arguments to reduce the bound to $\lceil \frac{n}{k} \rceil + 304$. Hoàng and Reed [7] showed that the conjecture is true for $k \leq 5$. Note that Conjecture 1, if true, implies that any k-regular digraph on n vertices has a cycle of length at most $\frac{2n}{k+1}$ (as proved in [5].)

Bermond and Thomassen [2] conjectured that any digraph with minimum outdegree k has at least $k/2$ vertex-disjoint cycles. Thomassen [13] proved that such a digraph has at least r vertex disjoint cycles if $k \geq (r+1)!$. In Section 3, we improve this to a linear bound for graphs with minimum degree k and maximum degree $2k$.

Let us say that a directed graph G has the *cycle-packing property* if the maximum size of a collection of edge-disjoint cycles equals the minimum size of a set of edges whose removal leaves an acyclic graph. The following proposition shows that Conjecture 1 is true for any such graph.

Proposition: *If G is a directed graph with no parallel edges and minimum outdegree k, and $S \subseteq E(G)$ meets every cycle in G, then $|S| \geq \binom{k+1}{2}$.*

Proof: Since $G - S$ is acyclic there is an ordering $v_1, v_2, \ldots, v_n$ of its vertices so that for every directed edge $v_i v_j$ of $G - S$, $i < j$. It follows that v_{n-j} has at least $k - j$ outedges in S, for all $0 \leq j < k$, implying the desired result. $\square$

Lucchesi and Younger [8] showed in 1978 that any planar digraph has the cycle-packing property. This result has recently been extended to Eulerian flat digraphs. A graph is *flat* if it can be embedded in R^3 so that each cycle bounds a disc disjoint from the rest of the graph; and a digraph is *flat* if the underlying undirected graph is. Examples of flat graphs include the *apex* graphs, that is graphs G such that $G \backslash v$ is planar for some vertex v. Seymour [12] shows that any Eulerian flat digraph has the cycle-packing property.

Unfortunately, these results have limited application here. If each outdegree in G is at least k and G has no 2-cycles, then G is not planar if $k > 2$, and G is not flat if $k > 3$.

Younger has conjectured that for any $r \geq 1$ there exist (least) integers $f(r)$ (resp. $g(r)$) such that every digraph has either a set of r edge- (resp. vertex-) disjoint cycles or a set of $f(r)$ (resp. $g(r)$) edges (resp. vertices) which meets every cycle. Soares pointed out that if $f(r), g(r)$ exist then they must be equal. McCuaig [9] showed $f(2) = 3$. For $r > 2$, $f(r)$ is not known to exist, but Alon and Seymour (see [11]) observed that if it exists then $f(r) = \Omega(r \log r)$, whereas Seymour proved in [11] that if a digraph does not have a "fractional" packing of directed cycles of value greater than k then one can delete $O(k \log k \log \log k)$ of its edges and obtain an acyclic digraph.

2 A Linear Bound

Proof of Theorem 1: We shall, in fact, show more strongly that if G is an Eulerian directed graph with no parallel edges and with minimum degree k, then G contains a collection of $5k/2 - 2$ edge-disjoint cycles. Let x be any vertex of degree k. Clearly we can find k edge-disjoint directed cycles through x. Choose a set of k such cycles $\mathcal{C} = C_1, \ldots, C_k$ such that the sum of the lengths is minimum.

Let D be the graph formed from all the arcs in $\mathcal{C}$.

Claim: The union of all the arcs of D not incident with x in $\mathcal{C}$ gives an acyclic graph H, and so generates a partial order $\mathcal{P}(\mathcal{C})$. Further, if $u < v$ under $\mathcal{P}$ then G has no $u \to v$ edge outside of $\mathcal{C}$.

Proof of Claim: Note that D is Eulerian, and x has degree k in D. Further, any such graph yields a collection of k edge-disjoint directed cycles through x. Thus, since $\mathcal{C}$ was chosen to minimize the number of arcs in the graph D, H must be acyclic, and similarly the second sentence in the claim follows. □

Note that as G has no multiple edges, any minimal element of $\mathcal{P}$ lies in exactly one cycle of $\mathcal{C}$. Note further that any two vertices, other than x, each of which lies in more than $\frac{k}{2}$ members of $\mathcal{C}$, are comparable, as they must lie on a common cycle. Hence, there must be some minimal element x_2 of $\mathcal{P}$ which is less than all such vertices.

Let G_2 be the graph formed by deleting the edges of $\mathcal{C}$ from G. Since x_2 lies in exactly one member of $\mathcal{C}$, it has degree at least $k-1$ in G_2. Therefore we can choose a set of $k-1$ edge-disjoint cycles in G_2, each passing through

x_2. Again, choose a set whose total length is minimum, and let $\mathcal{P}_2$ be the partial order that it induces. Remove its edges, leaving G_3.

Let x_3 be any minimal element in $\mathcal{P}_2$, and note that there is an edge from x_2 to x_3. By our Claim, G_2 has no edges from x_2 to any vertex which lies in more than $\frac{k}{2}$ members of $\mathcal{C}$, and so x_3 has degree at least $\frac{k}{2}$ in G_2. Also, x_3 lies in exactly one of the second set of cycles, and so x_3 has degree at least $\frac{k}{2} - 1$ in G_3. Thus we can find $\frac{k}{2} - 1$ edge-disjoint cycles in G_3, proving the theorem. $\square$

3 A Quadratic Bound

Note that if it were true that any Eulerian directed graph with minimum degree k has a collection of k vertex-disjoint cycles, then Conjecture 1 would follow. Unfortunately, this is not the case, for example with C_n^k where n is not a multiple of k or with the complete directed graph on $k + 1 > 2$ vertices. However, we can prove a weaker result which is enough to give us the quadratic bound of Theorem 2, and which may be interesting in its own right:

Lemma 1: *If G is a directed graph with no parallel edges, and with minimum degree at least $k \geq 1$ and maximum degree at most $2k$, then the vertices of G may be coloured with at least $k/2^{16}$ colours (each used) in such a way that for each colour, the corresponding induced subgraph H has all vertex indegrees and outdegrees in an interval $[a, 4a]$ where $a \geq 1$.*

Before presenting the proof of this lemma, we will see that it implies Theorem 2:

Proof of Theorem 2: We set $G_k = G$, and recursively define G_j for $\lceil \frac{k}{2} \rceil \leq j \leq k$. For each j, we can apply Lemma 1 to find a collection $\mathcal{C}_j$ of at least $j/2^{16}$ vertex disjoint cycles in G_j, and then define G_{j-1} as the graph obtained by deleting the edges of $\mathcal{C}_j$ from G_j. Now $\mathcal{C} = \cup_{j=\lceil \frac{k}{2} \rceil}^{k} \mathcal{C}_j$ is a collection of edge-disjoint cycles in G, where:

$$
\begin{aligned}
|\mathcal{C}| &\geq \sum_{j=\lceil \frac{k}{2} \rceil}^{k} \frac{j}{2^{16}} \\
&\geq \frac{3}{2^{19}} k^2
\end{aligned}
$$

□

Note that if the conjecture of Bermond and Thomassen discussed in Section 1 holds, then the value of ϵ in Theorem 2 can be raised to $1/4$.

The proof of Lemma 1 applies a method similar to the one used in [1] and makes use of the Lovász Local Lemma, which we state here in its symmetric case.

The Lovász Local Lemma [6]: *Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space, such that $\mathbf{Pr}(A_i) \leq p$ for each $1 \leq i \leq n$. Suppose that each event A_i is mutually independent of a set of all other events but at most d. If $\mathrm{e}p(d+1) \leq 1$, then $\mathbf{Pr}(\cap_{i=1}^{n} \overline{A_i}) > 0$.*

We use this to prove:

Lemma 2: *Suppose that H is a directed graph with no parallel edges, and with minimum degree x and maximum degree y, where $x \geq 1000$, and $y \leq 4x$. Then the vertices of H can be coloured red and blue such that for any vertex $v \in V(H)$, the number of red outneighbours of v lies in the interval $[\delta_H^+(v)/2 - \delta_H^+(v)^{2/3}, \delta_H^+(v)/2 + \delta_H^+(v)^{2/3}]$ (and thus so also does the number* *of blue ones), and similarly the number of red inneighbours of v lies in the interval* $[\delta_H^-(v)/2 - \delta_H^-(v)^{2/3}, \delta_H^-(v)/2 + \delta_H^-(v)^{2/3}]$.

Proof: Colour each vertex of H either red or blue, making each choice independently and uniformly at random. For each v, let A_v^+ be the event that the number of red outneighbours of v does *not* lie in the interval $[\delta_H^+(v)/2 - \delta_H^+(v)^{2/3}, \delta_H^+(v)/2 + \delta_H^+(v)^{2/3}]$, and define A_v^- similarly.

For each v,

$$\begin{aligned} \mathbf{Pr}(A_v^+) &\leq 2\mathrm{e}^{-2\delta_H^+(v)^{1/3}} \\ \mathbf{Pr}(A_v^-) &\leq 2\mathrm{e}^{-2\delta_H^-(v)^{1/3}}. \end{aligned}$$

Each of these probabilities is bounded above by $2\mathrm{e}^{-2x^{1/3}}$.

Furthermore, for each v, A_v^- is mutually independent of all but at most $\sum_{u \in N^-(x)}(\delta^+(u) + \delta^-(u) - 1)$ other events, and A_v^+ is mutually independent of all but at most $\sum_{u \in N^+(x)}(\delta^+(u) + \delta^-(u) - 1)$ other events. Both sums are at most $32x^2 - 1$.

Now, for $x \geq 1000$, $64x^2\mathrm{e}^{1-2x^{1/3}} < 1$. Therefore, by the Lovász Local Lemma, $\mathbf{Pr}(\cap_{v \in V(H)}(\overline{A_i^+} \cap \overline{A_i^-})) > 0$, and so there must be at least one satisfactory 2-colouring. □

We now use Lemma 2 to prove Lemma 1:

Proof of Lemma 1: Let $c = 15$ and note that $2^{1-c/3}(2^{1/3} - 1)^{-1} \leq \ln \frac{4}{3} \leq \frac{1}{3}$, and hence $1 - 2^{1-\frac{c}{3}}(2^{\frac{1}{3}} - 1)^{-1} \geq \frac{2}{3}$, and $\frac{2}{3}2^c \geq 1000$.

We shall see that, if $r = \lfloor \log_2 k \rfloor - c$, then G can be coloured as required with $2^r \geq 2^{-(c+1)}k$ colours. Clearly, we can assume $k \geq 2^{16}$, and so $r \geq 1$.

Let $f(x) = \frac{1}{2}x - x^{2/3}$, for $x \geq 1$. Let $z \geq k$, let $x_0 = z$, and $x_{i+1} = f(x_i)$ for $i = 1, 2, ..$ while $x_i \geq 1$. Clearly the x_i are decreasing and $x_i \leq 2^{-i}z$ while x_i is defined. Let $1 \leq j \leq r$ be such that $x_{j-1} \geq 1$, so that x_j is defined. Then

$$\begin{aligned}
x_j &= 2^{-1}x_{j-1} - x_{j-1}^{\frac{2}{3}} \\
&= 2^{-j}z - \sum_{i=1}^{j} 2^{-i+1}x_{j-i}^{\frac{2}{3}} \\
&\geq 2^{-j}z - \sum_{i=1}^{j} 2^{-i+1}(2^{-(j-i)}z)^{\frac{2}{3}} \\
&= 2^{-j}z - 2(2^{-j}z)^{\frac{2}{3}}(\sum_{i=1}^{j} 2^{-i/3}) \\
&\geq 2^{-j}z - 2(2^{-j}z)^{\frac{2}{3}}(2^{1/3} - 1)^{-1} \\
&= (2^{-j}z)(1 - 2(2^{-j}z)^{-\frac{1}{3}}(2^{1/3} - 1)^{-1}) \\
&\geq (2^{-j}z)(1 - (2^{1-\frac{c}{3}})(2^{1/3} - 1)^{-1}) \\
&\geq \frac{2}{3}(2^{-j}z) \\
&\geq \frac{2}{3}2^c\, 2^{r-j} \\
&\geq 1000\, 2^{r-j}.
\end{aligned}$$

Thus each x_j for $j = 1, .., r$ is defined, and satisfies $x_j \geq \frac{2}{3}(2^{-j}z) \geq 1000\, 2^{r-j}$.

Now let $g(x) = \frac{1}{2}x + x^{2/3}$, for $x \geq 0$. Let $y_0 = z$, and $y_{i+1} = g(y_i)$ for $i = 1, 2, ..$ Clearly we have $y_i \geq 2^{-i}z$. Let $1 \leq j \leq r$. Then

$$\begin{aligned}
y_j &= 2^{-1}y_{j-1}(1 + 2y_{j-1}^{-1/3}) \\
&= (2^{-j}z)\prod_{i=0}^{j-1}(1 + 2y_i^{-1/3})
\end{aligned}$$

$$
\begin{aligned}
&\leq (2^{-j}z)\exp(\sum_{i=0}^{j-1} 2y_i^{-1/3}) \\
&\leq (2^{-j}z)\exp(2z^{-1/3}\sum_{i=0}^{j-1} 2^{i/3}) \\
&\leq (2^{-j}z)\exp(2(2^{-j}z)^{-1/3}(2^{1/3}-1)^{-1}) \\
&\leq (2^{-j}z)\exp(2^{1-c/3}(2^{1/3}-1)^{-1}) \\
&\leq \frac{4}{3}(2^{-j}z).
\end{aligned}
$$

Thus each y_j for $j = 1, .., r$ satisfies $y_j \leq \frac{4}{3}(2^{-j}z)$.

Let us use Lemma 2 to 2-colour G, then to 2-colour each of the subgraphs induced by the colour classes, and so on. Suppose that we have performed this for j levels where $0 \leq j \leq r$, and let H be one of the 2^j corresponding induced subgraphs of G. Let u and v be any two vertices of H. By the above we see that $d_H^+(u) \geq \frac{2}{3}2^{-j}d_G^+(u) \geq \frac{2}{3}2^{-j}k \geq 1000\,2^{r-j}$, and $d_H^+(v) \leq \frac{4}{3}2^{-j}d_G^+(v) \leq \frac{4}{3}2^{-j}(2k)$; and there is a similar result for indegrees. Thus if H has minimum degree x and maximum degree y then $y \leq 4x$ and $x \geq 10002^{r-j}$. Hence, if $j < r$ we may continue to apply Lemma 2 to colour for one more level, and the lemma follows (with $a = \frac{2}{3}2^{-r}k$). $\square$

Remark: Using a more straightforward application of the Lovász Local Lemma (to repeatedly find vertex disjoint cycles as before, but without the iterated splitting procedure), one can also get a nearly quadratic bound with a more reasonable constant:

Theorem 3: *If G is a k-regular directed graph with no parallel edges and with $k \geq 2$, then G contains a collection of at least $k^2/8\ln k$ edge-disjoint cycles.*

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