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# On Total Colourings of Graphs

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We show that as $n \to \infty$ the proportion of graphs on vertices $1, 2, ..., n$ with total chromatic number $\chi'' > \Delta + 1$ is very small; and the proportion with $\chi'' > \Delta + 2$ is very very small. Here $\Delta$ denotes the maximum vertex degree. We also give an easy new deterministic upper bound on $\chi''$ (proved randomly). © 1993 Academic Press, Inc.

## 1. INTRODUCTION

Our interest here is in total colourings of graphs, but it is convenient to discuss edge colourings briefly first.

An *edge colouring* of a (simple) graph $G$ is an assignment of colours to the edges so that no two incident edges receive the same colour. The *edge chromatic number* (or chromatic index) $\chi'(G)$ is the least number of colours in an edge colouring of $G$. Vizing's theorem [9] states that $\chi' = \Delta$ or $\Delta + 1$, where $\Delta = \Delta(G)$ denotes the maximum degree of a vertex in $G$.

Is it rare for $\chi'$ to be forced above the trivial lower bound $\Delta$? Let $p'_n$ be the proportion of graphs on vertices $1, 2, ..., n$ with $\chi' > \Delta$. Erdős and Wilson [4] showed that $p'_n \to 0$ as $n \to \infty$. This result was much strengthened by Frieze, Jackson, McDiarmid, and Reed [5], who showed that

$$n^{-(1/2 + o(1))n} \leqslant p'_n \leqslant n^{-(1/8 + o(1))n} \qquad \text{as} \quad n \to \infty.$$

A *total colouring* of a graph $G$ is an assignment of colours to the vertices and edges of $G$ so that no two adjacent or incident elements receive the



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same colour. The *total chromatic number* $\chi''(G)$ is the least number of colours in such a colouring. The total colouring conjecture of Bezhad [1] and Vizing [10] asserts that always $\chi'' = \Delta + 1$ or $\Delta + 2$. For a recent survey on total colouring see Chetwynd [3].

Is it rare for $\chi''$ to be forced above the trivial lower bound $\Delta + 1$? Let $p''_n$ be the proportion of graphs on vertices $1, 2, \ldots, n$ with $\chi'' \geqslant \Delta + 2$. It was shown in the survey paper [8] that $p''_n \to 0$ as $n \to \infty$. For our first result here (see Theorem 2.1 below) we adapt the argument in [5] concerning $p'_n$ to show that

$$p''_n \leqslant n^{-(1/8 + o(1))n} \quad \text{as} \quad n \to \infty.$$

We do not know if this bound is of the right order. Indeed we have no non-trivial lower bound for $p''_n$.

Our second result (see Theorem 2.2 below) is that the corresponding proportion of graphs with $\chi'' > \Delta + 2$ is $o(c^{n^2})$ for some $0 < c < 1$. This upper bound is very small, but we would of course prefer it to be zero!

Our third and last result is rather different. It is a deterministic bound on $\chi''(G)$, but it is proved by (very elementary) random methods.

THEOREM 1.1. *If $G$ is a graph with $n$ vertices and $k$ is an integer with $k! \geqslant n$ then*

$$\chi''(G) \leqslant \chi'(G) + k + 1;$$

*and so as $n \to \infty$*

$$\chi'' \leqslant \Delta + O(\log n / \log \log n).$$

This result is stronger than previously known upper bounds (see [6]) only if $\Delta$ is large, say $\Delta \geqslant (\log n)^3$. Since Theorem 1.1 may be proved quickly and easily, let us do so now.

*Proof of Theorem* 1.1. We may assume that $G$ is not complete (since complete graphs satisfy $\chi'' \leqslant \Delta + 2$) and that $G$ is connected. Let $q = \chi'(G)$. By Brook's theorem, $G$ has a vertex colouring using $q$ colours, with collection of stable sets $\mathscr{S} = \{S_1, \ldots, S_q\}$. Let $\mathscr{M} = \{M_1, \ldots, M_q\}$ be the collection of matchings in an edge colouring of $G$ using $q$ colours. We may assume also that $2 \leqslant k \leqslant q - 1$; other cases are easy.

Given a bijection $\pi$ from $\mathscr{M}$ to $\mathscr{S}$, let the "rejection graph" $G'$ be the subgraph of $G$ containing those edges $\{x, y\}$ of $G$ such that, if $M$ is the matching in $\mathscr{M}$ containing the edge $\{x, y\}$, then either $x$ or $y$ is in the stable set $\pi(M)$. Then clearly

$$\chi''(G) \leqslant q + \chi'(G') \leqslant q + 1 + \Delta(G').$$

We shall show that for some bijection $\pi$ from $\mathcal{M}$ to $\mathcal{S}$ we have $\Delta(G')\leqslant k$, by considering a random bijection with all $q!$ equally likely.

Consider a vertex $v$ in $G$, with set $W$ of at least $k+1$ neighbours in $G$. Let $\mathcal{C}$ denote the collection of sets $W'\subseteq W$ with $|W'|=k$ such that no two vertices in $W'$ have the same colour in our vertex colouring. Also, for each set $W'\in\mathcal{C}$, let $A(W')$ be the event that for each vertex $w\in W'$ the matching $M\in\mathcal{M}$ containing the edge $\{v,w\}$ is mapped to the stable set $S\in\mathcal{S}$ containing $w$: observe that

$$P(A(W'))=\left(\prod_{i=0}^{k-1}(q-i)\right)^{-1}=\frac{(q-k)!}{q!}.$$

Let $d'(v)$ denote the degree of $v$ in $G'$. If $d'(v)\geqslant k+1$ then the event $A(W')$ must occur for at least one $W'\in\mathcal{C}$. Also of course $|\mathcal{C}|\leqslant\binom{|W|}{k}$ so

$$P\{d'(v)\geqslant k+1\}\leqslant\binom{|W|}{k}\frac{(q-k)!}{q!}.$$

Further if $|\mathcal{C}|=\binom{|W|}{k}$ then all the vertices in $W$ have distinct colours (since $k\geqslant 2$) and there is a positive probability that more than one of the events $A(W')$ will occur (since $q\geqslant k+1$); and hence the last inequality is strict. Thus we see that

$$\begin{aligned}P\{d'(v)\geqslant k+1\}&<\binom{|W|}{k}\frac{(q-k)!}{q!}\\&\leqslant\binom{\Delta}{k}\frac{(\Delta-k)!}{\Delta!}=\frac{1}{k!},\end{aligned}$$

since $|W|\leqslant\Delta$ and $q\geqslant\Delta$. But now we have

$$P(\Delta(G')\geqslant k+1)<n/k!\leqslant 1$$

since $k!\geqslant n$. So for some bijection $\pi$, $\Delta(G')\leqslant k$, as required. ∎

The three results introduced above (namely Theorems 2.1, 2.2, and 1.1) were first announced in the survey paper [8] with sketch proofs.

## 2. Random Graphs

The random graph $G_{n,p}$ has vertices $1, 2, ..., n$ and the $\binom{n}{2}$ possible edges appear independently with probability $p$. We shall restrict our attention here to constant $p$. The case $p=\frac{1}{2}$ corresponds to proportions as discussed in the first section. The following two theorems are our main results.

THEOREM 2.1. *Let $p$ and $c$ be constants with $0<p<1$ and $0<c<\frac{1}{3}$, $c<p/2$. Then*

$$P\{\chi''(G_{n,p})>\Delta+1\}=o(n^{-cn/2}).$$

THEOREM 2.2. *Let $p$ be a constant, $0<p<1$. Then there is a constant $c$, $0<c<1$, such that*

$$P(\chi''(G_{n,p})>\Delta+2)=o(c^{n^2}).$$

We prove Theorem 2.1 in the next section and finally prove Theorem 2.2. For both proofs we shall use the following lemma. Recall that the *stability number* $\alpha(G)$ of a graph $G$ is the largest size of a stable set of vertices; and the *achromatic number* $\psi(G)$ *of* $G$ is the largest number of colours in a (proper) vertex colouring such that for each pair of colours some vertex of the first colour and some vertex of the second colour are adjacent. The usual behaviour of $\psi(G_{n,p})$ [7] and of $\alpha(G_{n,p})$ (see, for example, [2]) is well known. We are interested here in extreme random behaviour.

LEMMA 2.3. *Let $0<p<1$ and $\varepsilon>0$. Then there exists $c$, $0<c<1$, such that*

$$P\{\alpha(G_{n,p})\geqslant \varepsilon n\}=o(c^{n^2}),$$

*and*

$$P\{\psi(G_{n,p})\geqslant \varepsilon n\}=o(c^{n^2}).$$

*Proof.* Let $q=1-p$, and let $k=k(n)=\lceil \varepsilon n\rceil$. Then

$$\begin{aligned}P\{\alpha(G_{n,p})\geqslant k\}&\leqslant \binom{n}{k} q^{\binom{k}{2}}\\&=\exp\left\{-\frac{\varepsilon^2}{2}\left(\log\frac{1}{q}\right)n^2+O(n)\right\}.\end{aligned}$$

To handle $\psi(G_{n,p})$, consider a partition of the vertex set $\{1, ..., n\}$ into $k$ sets, of sizes $n_1, ..., n_k$. Let $A$ be the event that for each pair of sets in the partition, some vertex in the first set and some vertex in the second set are adjacent. Then

$$\begin{aligned}P(A)=\prod_{i<j}(1-q^{n_in_j})&\leqslant \exp\left\{-\sum_{i<j} q^{n_in_j}\right\}\\&\leqslant \exp\left\{-\binom{k}{2} q^{n^2/k^2}\right\}.\end{aligned}$$

Hence

$$P\{\psi(G_{n,p}) \geq \varepsilon n\} \leq n^n \exp\left\{-\binom{k}{2} q^{n^2/k^2}\right\}$$
$$= o(c^{n^2}) \qquad \text{if} \quad c > \exp(-\tfrac{1}{2}\varepsilon^2 q^{1/\varepsilon^2}).$$

## 3. Proof of Theorem 2.1

We prove Theorem 2.1 by analysing an algorithm that attempts to total colour in $\Delta + 1$ colours and rarely fails. More specifically we describe a class of "good" graphs such that the algorithm always works on such graphs, and the probability that the random graph $G_{n,p}$ is not good is very small. We assume that the reader is familiar with [5].

Let us first describe the good graphs. We want the edges to be distributed reasonably evenly throughout the graph, we want the number of vertices of any given degree not to be too large, and we want to be able to vertex colour suitably.

For $0 < p < 1$ and $0 < \varepsilon < \min(p, 1-p)$ call a graph $G_n$ with $n$ vertices $(p, \varepsilon)$*-uniform* if

$$\text{(i)} \qquad A \subseteq V(G), \qquad |A| \geq \varepsilon n \Rightarrow \left| \frac{|E(A)|}{p\binom{|A|}{2}} - 1 \right| < \varepsilon;$$

$$\text{(ii)} \qquad A, B \subseteq V(G), \qquad A \cap B = \phi,\ |A|, |B| \geq \varepsilon n$$
$$\Rightarrow \left| \frac{|E(A, B)|}{p\,|A|\,|B|} - 1 \right| < \varepsilon.$$

Here $E(A)$ denotes the set of edges of $G_n$ with both end vertices in $A$, and $E(A, B)$ denotes the set of edges with one end in $A$ and the other in $B$.

Lemma 3.1 [5]. *Let $0 < p < 1$ and $\varepsilon > 0$. Then*

$$P\{G_{n,p} \textit{ is not } (p, \varepsilon)\textit{-uniform}\} = o(c^{n^2}),$$

*where $c = e^{-\varepsilon^2 p/7}$.*

Given also $0 < c < 1$ call a graph $(p, c, \varepsilon)$*-uniform* if it is $(p, \varepsilon)$-uniform and the number of vertices of any given degree is at most $cn$. Let us call $G_n$ $\varepsilon$*-well colourable* if it has a vertex colouring with at most $\varepsilon n$ colours in which each colour set is of size at most $\varepsilon n$. Finally, call $G_n$ $(p, c, \varepsilon)$*-good* if it is $(p, c, \varepsilon)$-uniform and $\varepsilon$-well colourable. By Lemma 2.3 above and Lemma 2 of [5] we have

LEMMA 3.2. *Given* $0<p<1$, $0<c<1$, *and* $\varepsilon>0$,

$$P\{G_{n,p} \text{ is not } (p, c, \varepsilon)\text{-good}\} = o(n^{-(1-o(1))cn/2}).$$

We next describe an algorithm that will always total colour a good graph in $\Delta+1$ colours, if $\varepsilon>0$ is sufficiently small and $n$ is sufficiently large. The algorithm is an adaptation of the algorithm in [5] for $\Delta$-edge-colouring.

Given a graph $G$ let $H$ denote both the set of vertices of maximum degree in $G$ and the subgraph induced by these vertices. The aim is to find pairwise disjoint vertex-edge colour sets $T_1^*, ..., T_l^*$ such that each vertex appears once, and if we form the graph $G'$ by deleting from $G$ all the edges that appear in the colour sets $T_i^*$ then we have the following three properties:

$\Delta(G') = \Delta(G) - l + 1$,

the set of vertices of maximum degree in $G'$ is also $H$,

$H$ induces a stable set in $G'$.

If the algorithm succeeds to this point then we may complete a total colouring in $\Delta+1$ colours by edge colouring $G'$ in $\Delta(G')$ colours.

It is convenient to let $T$ denote the set of vertices $v$ in $G$ with

$$pn - 2\varepsilon n \leqslant d_G(v) \leqslant \Delta - 2.$$

*The Algorithm*

Let us now describe the algorithm in some detail. It accepts as input a graph $G$ with $n$ vertices together with parameters $p, \varepsilon$; and has four steps.

*Step* 1 {colour the edges in $H$}.

Partition the edges of $H$ into $h = |H|$ matchings $T_1, ..., T_h$ in an "inequitable" way [5]; that is, $|T_i| \leqslant \theta n$ for $i \geqslant (p+\varepsilon)h$, where

$$\theta = \varepsilon/\min\{p+\varepsilon, 1-p-\varepsilon\}.$$

*Step* 2 {colour the vertices of $G$}.

Partition the vertices of $G$ into $k \leqslant \varepsilon n$ stable sets $T_{h+1}, ..., T_l$, where $l = h+k$, each of size at most $\varepsilon n$.

*Step* 3 {extend each $T_i$ to $T_i^*$ such that $T_i^*$ covers all but a few vertices of low degree}.

Let $G_0 = G$: for $i=1$ to $h$ let $L_i$ be the set of vertices covered by the matching $T_i$; and for $i = h+1, ..., l$ let $L_i$ be the stable set $T_i$.

For $i=1$ to $l$ do the following.

Form $G_i'$ from $G_{i-1}$ by deleting the edges in $H$ and the set $L_i \cup S_i$ of vertices, where $S_i$ is the set of vertices in $V \backslash L_i$ of degree less than $pn-(i-1)-2\varepsilon n$ in $G_{i-1}$. If $|V \backslash (L_i \cup S_i)|$ is odd then delete an extra vertex $v_i \in T$ distinct from any previously chosen $v_j$. Find a perfect matching $M_i$ of $G_i'$. Let $T_i^* = T_i \cup M_i$ and $G_i = G_{i-1} \backslash M_i$.

*Step* 4 {tidy up}.
Edge colour $G_l$ with $\Delta(G_l)$ colours.
Together with the $l$ colour sets $T_i^*$ this gives a total colouring of $G$ in $\Delta(G)+1$ colours.
Now let us analyse the above algorithm.

LEMMA 3.3 [5]. *Let* $0<p<1$ *and* $\varepsilon>0$. *Let* $G$ *be a* $(p, \varepsilon)$*-uniform graph with n vertices. Then, for n sufficiently large, at most* $\varepsilon n$ *vertices have degree at most* $\lceil (p-2\varepsilon) n \rceil$; *and* $\Delta(G) \geqslant (p-2\varepsilon) n+1$.

LEMMA 3.4. *Let* $0<p<1$ *and* $0<c<\min\{p/2, \frac{1}{3}\}$ *be given. If* $\varepsilon>0$ *is sufficiently small, n is sufficiently large, and G is a* $(p, c, \varepsilon)$*-good graph with n vertices then the algorithm succeeds in total colouring G with* $\Delta+1$ *colours.*

*Proof.* From our earlier discussion it suffices to prove that

(a) there is always a possible choice for the extra vertex $v_i$;

(b) each graph $G_i'$ has a perfect matching; and

(c) $\Delta(G_l)=\Delta(G)-l+1$, and $H$ is the set of vertices of maximum degree in $G_l$.

Observe first that $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$, that $S_1$ is the set of vertices of degree less than $pn-2\varepsilon n$ in $G$, and that $|S_1| \leqslant \varepsilon n$ for $n$ sufficiently large, by Lemma 3.3.

To prove (a), note that since in $G$ there are at most $cn$ vertices of any given degree we have from Lemma 3.3 that

$$|T| \geqslant n-2cn-\varepsilon n \geqslant (c+2\varepsilon) n \geqslant l+\varepsilon n$$

for $\varepsilon$ sufficiently small and $n$ sufficiently large. Thus there is always a vertex in $T-L_i$ distinct from any previously chosen $v_j$. (If $1 \leqslant i \leqslant h$ then $T$ and $L_i$ are disjoint.)

The proof of (3a) in [5] applies almost unchanged to yield (b) here, so it remains only to prove (c). But, for each vertex $v$ with $d_G(v)=\Delta$ or $\Delta-1$ we have $d_{G_l}(v)=d_G(v)-(l-1)$. For each vertex $v \in T$ we have $d_{G_l}(v)=d_G(v)-(l-1)$ or $d_G(v)-(l-2)$, and so $d_{G_l}(v) \leqslant \Delta-l$. Finally, consider $v \in S_1$, and note that $v$ can never have been chosen as the extra deleted vertex. Thus by our choice of the $S_i$ we have $d_{G_l}(v) < pn - 2\varepsilon n - l + 2 \leqslant \Delta - l + 1$, by Lemma 3.3. ∎

## 4. Proof of Theorem 2.2

In this section we sketch the proof of Theorem 2.2. We shall sometimes assert rather briskly that certain matchings exist: the reader who has got this far can easily supply the missing steps. If $0 < c < 1$ and some statement about $G_{n,p}$ holds with probability $1 - o(c^{n^2})$ we shall say that it holds with *very high probability*.

Let $H = \{v \in V: d_G(v) \geqslant \Delta - \varepsilon n\}$. We shall show that with very high probability, for some $k \leqslant \varepsilon n$ there is a vertex colouring $T_1, ..., T_k$ and pairwise disjoint matchings $M_1, ..., M_k$ such for each $i$, $M_i$ covers none of $T_i$ and all of $H - T_i$. Form $G'$ from $G$ by deleting all the edges in these matchings $M_i$. Then we shall have $\Delta(G') = \Delta(G) - k + 1$, and so

$$\chi''(G) \leqslant k + \chi'(G') \leqslant \Delta(G) + 2.$$

Let $k = \lfloor \varepsilon n \rfloor$ and let $S = \{v \in V: \ d_G(v) \leqslant (p - 2\varepsilon) n\}$. By Lemmas 3.1 and 3.3, with very high probability $\Delta > (p - \varepsilon) n$ and so $H \cap S = \phi$, and $|S| \leqslant \varepsilon n$.

*Case* (a)

There is a vertex $v_0 \in V - H - S$. By Lemma 2.3, with very high probability there is a vertex colouring $T_1, ..., T_k$ of $G$ with each $|T_i| \leqslant k$ and such that $v_0 \in T_1$ and $|V - S - T_1|$ is even. If $|V - S - T_i|$ is odd let $R_i = \{v_0\}$, and if not let $R_i = \phi$. Then with very high probability there is a family $M_1, ..., M_k$ of pairwise disjoint matchings in $G$ such that $M_i$ is a perfect matching on $V - S - T_i - R_i$ for each $i$ (much as with statement (b) in the proof of Lemma 3.4).

*Case* (b)

Suppose that Case (a) does not hold, so that $H = V - S$. Now with very high probability, $|H| \geqslant (1 - \varepsilon) n$ and $\Delta(G) \leqslant (p + 3\varepsilon) n$ since the number of edges is at least $\frac{1}{2} |H| (\Delta - \varepsilon n)$. We shall show that, with very high probability, there is a vertex colouring $T_1, ..., T_k$ with $|T_i| \leqslant \varepsilon n$ and $|H - T_i|$ even for each $i$; and then as above there will be a family $M_1, ..., M_k$ of pairwise disjoint matchings in $G$ such that $M_i$ is a perfect matching on $H - T_i$ for each $i$. Let us concentrate on the vertex colouring.

*Subcase.* $|H|$ even. In the subgraph $\bar{G}[H]$ of the complementary graph $\bar{G}$ induced on $H$, all vertex degrees are at least

$$(|H| - 1) - \Delta(G) \geqslant (1 - p - 4\varepsilon) n - 1 \geqslant \varepsilon n$$

with very high probability, if say $\varepsilon < (1-p)/5$. Then with very high probability $\bar{G}[H]$ has a perfect matching, and by Lemma 2.3 this extends to a vertex colouring $T_1, ..., T_k$ of $G$ with $|T_i| \leqslant \varepsilon n$ and $|H - T_i|$ even for each $i$.

*Subcase.* $|H|$ odd. With very high probability, for each $v \in V \backslash H$,

$$d_{\bar{G}}(v) > (n-1) - (\Delta - \varepsilon n) \geqslant (1 - p - 2\varepsilon)\, n - 1$$

and so (for $\varepsilon$ as above) in $\bar{G}$ there is a matching of $S$ into $H$ (the "empty matching" if $S$ is empty). By Lemma 2.3, with very high probability, this extends to a vertex colouring $T'_1, ..., T'_{k'}$ of $G$ with $k' \leqslant k/2$, and $T'_i \cap H \neq \phi$ and $|T'_i| \leqslant k$ for each $i$. By splitting each set $T'_i$ with $|H \cap T'_i|$ even we may then find a vertex colouring $T''_1, ..., T''_{k''}$ with $k'' \leqslant k$, and $|H - T''_i|$ even and $|T''_i| \leqslant k$ for each $i$.

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