CSC 2427/MAT 1500 Winter 2025, Assignment #2

Due: Mar 11 at 11:59 PM. Emailed to molloy@cs.toronto.edu

I will not answer any questions about the assignment after Mar 9.

You may consult the Alon-Spencer and Molloy-Reed books. You may also consult an introductory graph theory textbook, such as the one by West. You may not consult any other materials, including online sources.

You must solve problems 1, 2, 3 on your own. You can work with other students on problems 4,5. You must write your own solutions. List everyone that you worked with.

You must write complete proofs. We will grade a subset of the problems.

This assignment has two pages.

1. **10 pts** G is a cycle of length $25n$. Its vertices are partitioned into sets $V_1, ..., V_n$, each of size 25. Prove that G has an independent set containing at least one vertex from each V_i.

2. **10 pts** We have a graph G where every vertex v contains a list $L(v)$ of at least $L > 100$ colours. For any vertex v and colour c, define $T(v, c)$ to be the set of neighbours u of v with $c \in L(u)$.

   Prove that if for every v we have

   $$\sum_{c \in L(v)} |T(v, c)| \leq \frac{1}{10} L|L(v)|$$

   then we can properly colour the graph giving each vertex a colour from its list.

   Hint: For any edge uv and colour $c \in L(u) \cap L(v)$, define $A(uv, c)$ to be the event that u, v are both assigned c.

3. **20 pts** Prove there is some $\gamma > 0$ such that the following holds for every graph G with maximum degree Δ:

   If for every vertex $v \in G$, there are at most $.999\binom{\Delta}{2}$ edges joining two neighbours of v then $\chi(G) \leq (1 - \gamma)\Delta$.

4. **20 pts** In this problem you will prove there is some integer r_0 such that for every $r \geq r_0$ the following holds:

   Every r-regular graph with a Hamilton cycle has at least two Hamilton cycles.

   (This is conjectured to be true for every $r \geq 3$.)

   (a) Let G be any r-regular graph with Hamilton cycle H. Prove that we can (i) choose $A \subset V(G)$ which is an independent set in H (i.e. A does not contain any two vertices that are adjacent in H) and (ii) find a subgraph $T \subset G$ whose edges consist of:

   - all edges of H;
   - for every v that is adjacent in H to A, exactly one edge not in $E(H)$ that goes from v to a vertex of A.

   (b) Pick any edge xy of H, where $x \in A$. Let $\mathcal{P}$ denote the set of all Hamilton paths in T whose first two vertices are x then y and whose last vertex is not in A. (We'll view each Hamilton path as having a first and last vertex.) Show that this implies the last vertex must be adjacent in H to A.

   (c) $\mathcal{B}$ is a graph with vertex set $\mathcal{P}$ where $P_1, P_2 \in \mathcal{P}$ are adjacent if P_2 can be formed from P_1 by deleting an edge st from P_1 and then adding an edge joining either s or t to the last vertex of P_1. Show that every path in $\mathcal{P}$ has degree at most 2 in $\mathcal{B}$.

   (d) The path formed from H by deleting the other edge incident with x (i.e. the edge xy' where $y' \neq y$) is in $\mathcal{P}$ because A is an independent set and so $y' \notin A$. Argue that this path has degree 1 in $\mathcal{B}$. Then argue that there must be at least one more path with degree 1 in $\mathcal{B}$. Argue that you can create a second Hamilton cycle using that path.

   The proofs for parts b, c, d do not involve probabilistic arguments.

5. **25 pts** G is Δ-regular, and every vertex v of G has a list $L(v)$ of 2Δ colours. Prove that you can colour G from these lists such that no colour appears more than $C \log \Delta / \log \log \Delta$ times in any neighbourhood. C is a constant that you can choose to be as big as you wish.

   To get you started: Order the vertices $v_1, ..., v_n$. Process them one-at-a-time. At step i, assign to v_i a colour chosen uniformly from the colours of $L(v_i)$ that are not on any neighbours of v_i. Apply the Lopsided Local Lemma. Be very careful about analyzing conditioning - particularly conditioning on events involving vertices that come later in the process.