CSC 2427/MAT 1500 Winter 2025, Assignment #1

Due: Jan 26 at 11:59 PM. Emailed to molloy@cs.toronto.edu

I will not answer any questions about the assignment after Jan 23.

You may consult the Alon-Spencer and Molloy-Reed books. You may also consult an introductory graph theory textbook, such as the one by West. You may not consult any other materials, including online sources.

You can work with other students on problems 2, 3 6, 7. You must write your own solutions. List everyone that you worked with.

You must write complete proofs. We will grade a subset of the problems.

This assignment has two pages.

1. **10 pts**

   Recall: An r-colouring of a hypergraph is an assignment of colours to vertices such that no hyperedge is monochromatic. The chromatic number of a hypergraph is the smallest r such that it has an r-colouring.

   Prove that every k-uniform hypergraph with at most 4^{k-1} hyperedges has chromatic number at most four. (This is problem 1 from Alon-Spencer Chap 2.)

2. **15 pts** Prove that for any $k, r > 1$, there exists a k-uniform hypergraph with chromatic number at least r, such that no two hyperedges share more than one vertex.

3. **20 pts** H is a graph with fewer than n vertices. G is a graph with n vertices and t edges that does not contain any copy of H (i.e. a subgraph isomorphic to H). Prove that if $tk > n^2 \log_e n$ then it is possible to k-colour the edges of K_n (the complete graph on n vertices) so that there is no monochromatic copy of H. (This is problem 5 from Alon-Spencer Chap 2.)

   Hint: You might find it easier to think about this problem if you allow each edge to take multiple colours.

4. **20 pts** G is a bipartite graph with n vertices on each side. Prove that the list-chromatic number of G is at most $1 + \log_2 n$. (This is problem 9 from Alon-Spencer Chap 2.)

5. **20 pts** Let $\mathcal{F}$ be a family of subsets of $\{1, ..., n\}$ such that there is no pair $A, B \in \mathcal{F}$ with $A \subset B$.

   (a) Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.
   Hint: Let σ be a uniformly random permutation of $\{1, ..., n\}$ and define X to be the number of integers i such that the set $\{\sigma(1), \sigma(2), ..., \sigma(i)\}$ is in $\mathcal{F}$. Consider the expected value of X.

   (b) Show that the bound in (a) is tight by giving a set $\mathcal{F}$ with $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ for every n.

   (This is problem 7 from Alon-Spencer Chap 2.)

6. **20 pts** A K_k-minor of a graph G is a collection of k disjoint connected subgraphs $H_1, ..., H_k$ of G such that for every $1 \leq i < j \leq k$, there is an edge between H_i and H_j. (You may have seen a different definition of minor - with a bit of thought you will see that it is equivalent to this definition.)

   A K_k-pseudo-minor of a graph G is a collection of k disjoint (not neccesarily connected) subgraphs $H_1, ..., H_k$ of G such that for every $1 \leq i < j \leq k$, the subgraph induced by $H_i \cup H_j$ is connected.

   The following conjecture stood open for roughly 10 years:

   Conjecture: Every graph that contains a K_k-pseudo-minor contains a K_k-minor.

   Disprove the conjecture when k is a sufficiently large constant.

   Hint: Consider a graph consisting of k pairs of vertices $(a_1, b_1), ..., (a_k, b_k)$ such that for each $1 \leq i < j \leq k$, three of the four possible edges between (a_i, b_i) and (a_j, b_j) are present.

7. **25 pts** G is a bipartite graph with bipartition A, B with $|A| \geq |B|$. G has minimum degree d. You will prove that the list-chromatic number of G is at least $(\frac{1}{2} - o(1))\log_2 d$.

   Let s be an integer and set $d = \_\_\ 2^{2s}$. Part of your job is to fill in the blank with some function of s which is small enough that $s = (\frac{1}{2} - o(1))\log_2 d$.

   You must show that it is possible to give every vertex a list of size s whose colours all come from $S = \{1, 2, 3, ..., s^2\}$ such that G does not have any proper colouring from these lists.

   Because this is an asymptotic bound, you can assume s to be at least as big as any large constant of your choice. You don't have to name this constant; instead you can say s is big enough that certain bounds hold. Eg s is big enough that $\sqrt{s} > 100(\log s)^{10}$. (This particular bound is not intended to be useful; it's just an example.)

   (a) The first step is to choose lists for *some* of the vertices in B. Each $u \in B$ is selected to receive a list L_u with probability $1/\sqrt{d}$. Let B' denote the selected vertices.

   We say that a vertex $v \in A$ is *surrounded* if for every subset $T \subset S$ with $|T| = \lceil \frac{1}{2}|S| \rceil$, v has at least one neighbour $u \in B$ with a list $L_u \subset T$. Prove that it is possible to choose lists for the vertices in B' so that at least half the vertices in A are surrounded.

   (b) Fix a set of lists on B' so that at least half of A is surrounded. .

   Colour every vertex of B' with one of the colours from its list; we call this a *legal* colouring of B'. Prove that every surrounded vertex in A has at least $\lceil \frac{1}{2}|S| \rceil$ different colours appearing in its neighbourhood.

   (c) Prove that it is possible to assign lists to the vertices of A such that: For every legal colouring of B' there is at least one $v \in A$ such that every colour in v's list appears on a neighbour of v in B'.

   Conclude that the list-chromatic number of G is at least s.

   (d) Modify your proof so that it works even if G is not bipartite.