CSC 2427 Spring 2008, Assignment #2

Due: Mar 13 at the *beginning* of class.

You may consult the two texts that are recommended for this course. You may also consult an introductory graph theory text, such as the one by West. You may not consult any other materials.

You can work with other students on problems 1, 4 and 5. You must write your own solutions. List everyone that you worked with.

1. **10 pts** Suppose that G is the Δ-regular graph consisting of two D-cliques with vertex sets $\{x_1, ..., x_\Delta\}, \{y_1, ..., y_\Delta\}$ along with the edges $\{(x_1, y_1), ..., (x_\Delta, y_\Delta)\}$. We will choose a random partial Δ-colouring of G as follows:

   1) Choose two uniformly random permutations σ_x, σ_y of the colours $\{1, ..., \Delta\}$. For each i, assign the colour $\sigma_x(i)$ to x_i and assign the colour $\sigma_y(i)$ to y_i.

   2) Uncolour any vertex that is given the same colour as a neighbour.

   We define X to be the number of neighbours of v that retain their colours.

   (a) What is the expected value of X?

   (b) Prove that $Pr(|X - E(X)| > \Delta^{2/3}) < \Delta^{-10}$. (The constants "2/3" and "10" are somewhat arbitrary.)

   Hint: Consider selecting a uniformly random permutation σ by first choosing $\sigma(1)$, then $\sigma(2)$, etc.

2. **10 pts** You have a Δ-regular graph G. Each vertex has a list of Δ available colours. You form a partial list-colouring of G as follows:

   1) Give each vertex a uniformly random colour from its list.

   2) Uncolour any vertex that is given the same colour as a neighbour.

   For a particular vertex v, we define X_v to be the number of neighbours of v that retain their colours.

   (a) Why do Azuma's Inequality and Talagrand's Inequality not work well when applied directly to X_v?

   (b) Prove that $Pr(|X_v - E(X_v)| > \Delta^{2/3}) < \Delta^{-10}$.

   Hint: Let Y_v be the number of vertices in N_v that are either uncoloured in Step 2 or are assigned a colour that was also assigned to at least $\log \Delta$ other vertices in $N(v)$.

3. **10 pts** You are given a graph G of maximum degree Δ and you are also given an ordering of its vertices: $v_1, ..., v_n$. You give each vertex a uniformly random label from $\{1, ..., \Delta\}$. A vertex v_i *wins* if its label does not appear on any neighbour v_j with $j < i$. Let Y be the number of winners. Prove that $Pr(|Y - E(Y)| > \Delta^{2/3}) < \Delta^{-10}$.

4. **15 pts** Read Sections 9.1, 9.2 of *Random Graphs* by Janson, Luczak and Rucinski or Section 2.4 of *Random Graphs* by Bollobas to find the definition of the "Configuration Model" for generating random Δ-regular graphs, and to learn that for each constant Δ there is a constant $\epsilon > 0$ such that the probability of the configuration being simple and triangle-free is at least ϵ.

   (Come to see either me or Hamed if you are having a hard time understanding this model.)

   Then prove that for every Δ sufficiently large, there exists a triangle-free graph of maximum degree Δ that has chromatic number at least $C\Delta / \log \Delta$ for some constant C.

   This assignment has two pages.

5. **15 pts** Suppose that G is a graph with maximum degree Δ. Suppose further that there is some constant $\epsilon > 0$ such that for every vertex $v \in G$: there are at most $(1-\epsilon)\binom{\Delta}{2}$ edges that join two neighbours of v.

   (a) Prove that there is some $\gamma = \gamma(\epsilon)$ such that $\chi(G) \leq (1-\gamma)\Delta$.

   (b) Prove that there is some $\gamma = \gamma(\epsilon)$ such that the *list chromatic number* of G is at most $(1-\gamma)\Delta$.

   Remarks: this is a slightly weaker version of Exercise 10.2 from the Molloy-Reed book. Of course, (b) implies (a); so if you provide a complete solution for (b) then you don't have to provide one for (a).