CSC 2427 Spring 2008, Assignment #1

Due: Feb 14 at the *beginning* of class.

You may consult the two texts that are recommended for this course. You may also consult an introductory graph theory text, such as the one by West. You may not consult any other materials.

You can work with other students on problems 1,3,5,7. You must write your own solutions. List everyone that you worked with.

1. **5 pts** Suppose that $\mathcal{X} = X_1, ..., X_m$ is a sequence of independent random trials. Suppose that $A_1, ..., A_n$ is a set of events where each A_i is determined by $F_i \subseteq \mathcal{X}$. Prove that if $F_i \cap (F_{i_1}, ..., F_{i_k}) = \emptyset$ then A_i is mutually independent of $\{A_{i_1}, ..., A_{i_k}\}$.

   For example, in the hypergraph colouring problem that we covered in class, X_i is the assignment of a colour to vertex v_i, A_j is the event that hyperedge e_j is monochromatic and $F_j = \{X_i | v_i \in e_j\}$.

2. **10 pts** Let $\mathcal{F}$ be a family of subsets of $\{1, ..., n\}$ with the property that there is no pair $A, B \in \mathcal{F}$ with $A \subset B$.

   (a) **9 pts** Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

   Hint: Let σ be a uniformly random permutation of $\{1, ..., n\}$ and define X to be the number of integers i such that the set $\{\sigma(1), \sigma(2), ..., \sigma(i)\}$ is in $\mathcal{F}$. Consider the expected value of X.

   (b) **1 pt** Show that the bound in (a) is tight by giving a set $\mathcal{F}$ with $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ for every n.

3. **10 pts** An r-colouring of a hypergraph is an assignment of colours to vertices such that no hyperedge is monochromatic. The chromatic number of a hypergraph is the smallest r such that it has an r-colouring.

   Prove that for any $k, r > 1$, there exists a k-uniform hypergraph with chromatic number at least r, such that no two hyperedges share more than two vertices.

   Remark: It is possible to ensure that no two hyperedges share more than *one* vertex, but that is not required for this assignment.

4. **10 pts** For $k, t\Delta > 1$ define $\ell = k\Delta^{1/(t-1)}/100t$. Prove that every k-uniform hypergraph with maximum degree Δ can be ℓ-coloured so that no colour appears at least t times in any hyperedge.

5. **10 pts** An adversary has coloured the edges of the complete graph on n vertices such that no colour appears more than $\frac{n}{100}$ times. Prove that there is a Hamilton cycle on which no colour appears twice.

6. **15 pts** Prove that for every Δ there exists $g = g(\Delta)$ such that the following is true:

   Every graph with maximum degree Δ and girth at least g can be edge-coloured using $\Delta + 2$ colours such that every cycle contains at least 3 colours.

   Hint: We know by Vizing's Theorem that every such graph can be edge-coloured with $\Delta + 1$ colours. Begin with such a colouring, and then change the colours of some of the edges to $\Delta + 2$.

This assignment has two pages.

7. **15 pts** A K_k-minor of a graph G is a collection of k disjoint connected subgraphs $H_1, ..., H_k$ of G such that for every $1 \leq i < j \leq k$, there is an edge between H_i and H_j. (You may have seen a different definition of minor - with a bit of thought you will see that it is equivalent to this definition.)

   A K_k-pseudo-minor of a graph G is a collection of k disjoint (not neccesarily connected) subgraphs $H_1, ..., H_k$ of G such that for every $1 \leq i < j \leq k$, the subgraph induced by $H_i \cup H_j$ is connected.

   The following conjecture stood open for roughly 10 years:

   Conjecture: Every graph that contains a K_k-pseudo-minor contains a K_k-minor.

   (a) **2 pts** Prove that every graph that contains a K_k-pseudo-minor contains a $K_{\lfloor k/2 \rfloor}$-minor.

   (b) **13 pts** Disprove the conjecture.
   Hint: Consider a graph consisting of k pairs of vertices $(a_1, b_1), ..., (a_k, b_k)$ such that for each $1 \leq i < j \leq k$, three of the four possible edges between (a_i, b_i) and (a_j, b_j) are present.