We don't even know the answer to the following, for any small value of $k \geq 3$:

Question 5.7 *Consider $k \geq 3$. Run the random graph process until the first k-core appears. Is the resulting graph a.s. not k-list-colourable?*

I think that the answer to this question is almost certainly NO for every $k \geq 3$. In fact, the answer to the following question is probably YES:

Question 5.8 *Consider $k \geq 3$. Run the random graph process until the graph is not k-list-colourable. Is the resulting graph a.s. not k-colourable?*

So, I think that the thresholds for k-list-colourability and k-colourability are equal.

We know that the answer to Question 5.7 is NO for large values of k, since Alon et al[**?**] proved that for large k and $c < \frac{1}{100} k \ln k$, $G_{n,M=cn}$ is a.s. k-list colourable.

6 Asymptotics

As we saw in sections 3.2,4.2, we know that $(\frac{1}{2}+o(1))\frac{2^k}{k} < d_k \leq \ln 2 / \ln(2^k/(2^k - 1)) < \ln 2 \times 2^k$. Subsequent efforts have improved the constant term in this lower bound, but no one has yet been able to improve the asymptotic order of either bound. It would be very interesting to get asymptotic upper and lower bounds with the same order of magnitude.

Question 6.1 *What is the asymptotic order of d_k?*

Recall that every lower bound on d_3 established in section 4.2 was obtained by analysing an algorithm which will a.s. produce a satisfying assignment for $F^3_{n,M=cn}$. Each of those algorithms, when extended to general k, will a.s. fail for $c > O(2^k/k)$. It is entirely plausible that $d_k = 2^k(\ln 2 + \text{o}(1))$ but that there is no efficient algorithm which will find a satisfying solution for $c > O(2^k/k)$.

Question 6.2 *Is there a constant $\epsilon > 0$ and a polytime algorithm P such that for all $k \geq 3$, a.s. P will find a satisfying solution for $F^k_{n,M=\epsilon\times d_k n}$?*

A similar situation is true for colourability. We know of no algorithm which can guarantee a k-colouring of $G_{n,M=cn}$ for $c > (\frac{1}{2}+\epsilon)k \ln k$. In fact, it may be the case that no such algorithm exists.

Question 6.3 *Is there a constant $\epsilon > 0$ and a polytime algorithm P such that for all $k \geq 3$, a.s. P will find a k-colouring of $G_{n,M=(\frac{1}{2}+\epsilon)k\ln k\times n}$?*

We don't even know of any efficient algorithm which will find a stable set large enough to form one of the colour classes in such a colouring.

Question 6.4 *Is there a constant $\epsilon > 0$ and a polytime algorithm P such that for all $k \geq 3$, a.s. P will find a stable set of size n/k in $G_{n,M=(\frac{1}{2}+\epsilon)k \ln k \times n}$?*

An analogous question can be asked for $G_{n,M}$ for much larger values of M. See [12] for some recent work on this problem.

We understand the asymptotics of c_k much better than those of d_k, thanks to the following theorem of Łuczak[44]:

Theorem 6.5 $c_k = (1 + \mathrm{o}(1))k \ln k$.

(Łuczak's theorem is actually stronger than the version which we present here, in that he allows for a more general range of p and he is more precise about the $o(1)$ term.)

Proof Outline We will work in the model $G_{n,p}$, and we will consider c to be a very large constant.

The asymptotic upper bound follows from Theorem 3.1 (as well as Theorem 3.2). Recall that this upper bound followed by analysing the size of the largest stable set in $G_{n,p}$. To prove the asymptotic lower bound, one must essentially prove that a.s. the graph can be coloured using k colour classes most of which are very close to the maximum size in the graph.

Given c, we define $a = \frac{n}{c}(\ln c - 30 \ln \ln c)$. Our goal will be to colour all but $2n/\ln^3 c$ of the vertices, using colour classes of size a. Similar calculations to those in the proof of Theorem 4.2 show that a.s. $G_{n,p=2c/n}$ has no subgraph of size at most $2n/\ln^3 c$ with average degree at least $c/\ln^2 c$. Therefore, the remaining vertices can be coloured with $c/\ln^2 c$ new colours. Thus, the total number of colours used in this colouring is less than $\frac{n}{a} + \frac{c}{\ln^2 c} = \frac{c}{\ln c}(1 + o(1))$. This proves that a.s. $\chi(G_{n,p=2c/n}) \leq k$ for $c \leq (1 + o(1))k \ln k$, thus proving the theorem.

The first, and most technically detailed, part of Łuczak's proof is to show that a.s. *many* of the vertices can be coloured using colour classes of size a. We omit the proof.

Lemma 6.6 *Consider any sufficiently large constant d, and any $a < a(d, n) = \frac{2n}{d}(\ln d - \ln \ln d)$. A.s. $G_{n,p=d/n}$ has a subgraph of size $n \ln^{-5} d$ which can be coloured with $n \ln^{-5} d/a$ colour classes, each of size a.*

(The statement of the lemma in [44] only says that many of the vertices can be coloured using colour classes of average size d, but Łuczak's proof shows that we can ensure every colour class has size a.)

He then applies what is known as the "expose-and-merge" technique, introduced in Matula[**?**] and in Matula and Kucera[**?**]. He examines $G_{n,p=2c/n}$ one small piece at a time, each time appealing to Lemma 6.6 to guarantee that he can find several appropriately large stable sets in that piece. He assigns colours to those stable sets and removes them from G and then proceeds to the next iteration. More specifically, he considers the following procedure:

1. Initially $V_1 = V(G)$ and $\mathcal{C} = \emptyset$.
2. For $i = 1$ to $\lceil \ln^{33} c - \ln^{30} c \rceil$ do the following:
   (a) Choose a random subset $H_i \subset V_i$ of size $n/\ln^{28} c$.
   (b) Take a colouring of $n \ln^{-33} c$ of the vertices of H_i using $n \ln^{-33} c/a$ colour classes of size a. Add the colour classes to $\mathcal{C}$.
   (c) Remove all coloured vertices from V_i to yield V_{i+1}.

When this procedure terminates, we let V^* be the set of all remaining uncoloured vertices. We know that (i) the colour classes of $\mathcal{C}$ all have size a, and (ii) $|V^*| = n - n \ln^{-33} c \times \lceil \ln^{33} c - \ln^{30} c \rceil \leq \frac{n}{\ln^3 c}$, as required.

The only thing that must be proven is that we will a.s. always be able to find the partial colouring needed in step 2b. Consider the first iteration. Expose the subgraph induced by H_1. It has the same distribution as $G_{n',p=c'/n'}$ where $n' = n/\ln^{28} c$ and $c' = 2c \times n'/n = 2c/\ln^{28} c$. Note that $a(c') > a$ and so Lemma 6.6 proves that the required partial colouring exists.

However, the same argument does not apply to H_2. The problem is that we have already exposed some of the edges between uncoloured vertices from H_1. Since H_2 will probably contain several of these vertices, we can no longer treat it as a "fresh" random graph.

Another way of looking at this is as follows: To determine our entire random graph, we can flip a sequence of $\binom{n}{2}$ coins, one for each pair of vertices, and each coin flip determines whether there is an edge between that pair. The order in which we make these coin flips doesn't matter, so long as we only make one flip for each pair. When we expose the graph induced by H_1, we make the appropriate coin flip for each pair of vertices in H_1. If some $u, v \in H_1$ also belong to H_2, then we cannot make a second coin flip for u, v.

The "expose-and-merge" technique allows us to overcome this problem. For the moment, we will ignore the issue of exposed edges, and treat each H_i as a fresh random graph $G_{n',p=c'/n'}$. In particular, if some pair of vertices $u, v \in H_i$ have already been "flipped" during a previous iteration, then we make a *fake flip* for them and pretend that the fake flip is the correct flip. Under this pretension, H_i is distributed exactly like $G_{n',p=c'/n'}$ and so the probability that step 2b fails for H_i is $o(1)$. Since there are less than $\ln^{33} c = O(1)$ iterations, a.s. every iteration succeeds.

But of course, we cheated. Quite possibly, several of the colour classes in $\mathcal{C}$ are not actually stable sets; they might contain pairs of vertices u, v for which the original flip indicated an edge, but which were assigned the same colour during a later iteration because the fake flip indicated no edge. So we add a final step to our procedure where we uncolour each such pair of vertices, and add them to V^*. Let Y be the number of such pairs. We wish to bound $\mathbf{Exp}(Y)$.

So consider any pair of vertices u, v and any pair of iterations $i < j$. Since $|V_i| \geq n/\ln^3 c$, the probability that u, v both lie in H_i is at most $\frac{1}{\ln^{25} c} \times \frac{1}{\ln^{25} c - 1}$.

The same is true for H_j. The probability that uv is exposed to be an edge in H_i is p. Recall that we treat H_j as a fresh random graph so, by symmetry, every pair of vertices is equally likely to receive the same colour. Thus, the probability that $u, v \in H_j$ both receive the same colour during iteration j is easily seen to be less than $\frac{\ln^{24} c}{c}$. Therefore,

$$\mathbf{Exp}(Y) \leq \binom{n}{2} \times \binom{\ln^{33} c}{2} \times p \times \left(\frac{1}{\ln^{25} c(\ln^{25} c - 1)}\right)^2 \times \frac{\ln^{24} c}{c} < \frac{1}{\ln^{10} c} n.$$

By Markov's Inequality, with probability at least $1 - \frac{1}{\ln^7 c} > \frac{1}{2}$ the number of such pairs is at most $\frac{n}{\ln^3 c}$. Thus, with probability at least $\frac{1}{2}$, $|V^*| \leq \frac{2n}{\ln^3 c}$ which implies that the chromatic number is at most $\frac{c}{\ln c}(1 + o(1))$. Therefore, by Theorem 5.1, a.s. $\chi(G_{n,p=2c/n}) \leq \frac{c}{\ln c}(1 + o(1)) + 4 = \frac{c}{\ln c}(1 + o(1))$. $\square$

We should remark that this procedure does not yield an efficient algorithm to find the colouring. The main problem is that the proof of Lemma 6.6 only establishes the existence of the partial colouring needed for step 2b, and does not provide an efficient way to find it. In fact (recall Question 6.4), there is no known efficient algorithm to find even one of the stable sets guaranteed by this lemma.

7 A new question

We close this survey with a new question of the same flavour as Bollobas' Question 4.1. This question is posed here by Achlioptas and Molloy.

Consider a graph G and any collection of k disjoint stable sets $V_1, ..., V_k$. We say that this collection is *bilinked* if for every pair V_i, V_j, the bipartite graph induced by $V_i \cup V_j$ is connected.

Question 7.1 *Consider a random graph process on n vertices and $k \geq 3$. Does the chromatic number of the graph jump from k to $k+1$ within $o(n)$ steps of the appearance of the first bilinked collection of k stable sets, other than a triangle?*

The motivation for this question is as follows.

Consider any bilinked collection, set $a_i = |V_i|$, and let B be the subgraph induced by $V_1 \cup ... \cup V_k$. There are at least $a_i + a_j - 1$ edges between V_i, V_j and so the total number of edges in B is at least $(k-1)(a_1 + ... + a_k) - \binom{k}{2}$. Since $|B| \geq k$, this is at least $\frac{k-1}{2}|B|$ and so B has average degree at least $k-1$. Furthermore, for $k = 3$ if B is not a triangle then $|B| \geq 4$ and this shows that the average degree in B is at least 2.5. Therefore, a straightforward argument along the lines of Theorem 4.2 shows that a.s. any bilinked collection of at least 3 stable sets in $G_{n,p=c/n}$ which is not a triangle, must contain at least ϵn vertices for some $\epsilon = \epsilon(c) > 0$.