CSC 2410 Final Exam

Dec 16, 2015

Write your solutions in the exam booklets provided. Put your name on every booklet.

Ask me if you don't understand a question.

You can use the West book. You can't use any other books or notes.

Unless stated otherwise, you can use as fact any theorem which I presented during the lectures. You can also use any theorems from the West book.

These problems all have reasonably short solutions. A solution that is too long might not get full marks, even if it is correct; for example, a proof with a very large number of cases. The grader will subjectively decide what constitutes "too long".

All graphs are simple, unless stated otherwise.

All graphs are simple except in problem 3

1. **(10 pts)** G is not perfect, but every induced subgraph of G is perfect. (We call this a *minimal imperfect graph.*) Prove that no clique of G is a vertex cutset.

   See Definition 8.1.1 on page 319. You may not use Theorem 8.1.29 or Corollary 8.1.30.

2. **(10 pts)** Prove that a planar embedded graph is bipartite iff every face is even.

3. **(15 pts)** You are given a sequence of integers $d_1 \geq d_2 \geq d_3 \geq ... \geq d_n$. Prove that there is a loopless multigraph (i.e. multiple edges are permitted but loops are not permitted) with degree sequence $d_1, ..., d_n$ iff both:

   (i) $\sum_{i=1}^{n} d_i$ is even; and

   (ii) $d_1 \leq d_2 + d_3 + ... + d_n$.

   This is Exercise 1.3.63. The hint on page 509 suggests an inductive proof, but there are other approaches.

4. **(15 pts)** G is a bipartite graph with bipartition (X, Y). Every vertex has degree at least 1. $d(x) \geq d(y)$ for every edge xy with $x \in X, y \in Y$. Prove that G has a matching that saturates X.

5. **(20 pts)** Prove that if G has girth at least 5 and is not a forest, then $\overline{G}$ has a Hamilton cycle. This is Exercise 7.2.25. Hint: If $\overline{G}$ does not satisfy Ore's condition, then what can you say about it?

6. **(20 pts)** Exercise 6.3.14 says:

   (*) If G is an embedded planar graph and every face has size 3 then G is 3-colourable iff G is Eulerian.

   You don't need to prove (*). You can use it for parts (a) and (b) below. Recall the characterization of *Eulerian graphs* from Theorem 1.2.26.

   Prove each of the following statements:

   (a) If G is an Eulerian planar embedded graph with minimum degree > 2, the outer face has size 5 and every other face has size 3, then $\chi(G) = 3$.

   (b) There is no Eulerian planar embedded graph with minimum degree > 2 where one face has size 5 and every other face has size 3.