CSC 2410 Final Exam

Apr 10, 2014

Write your solutions in the exam booklets provided. Put your name on every booklet.

Ask me if you don't understand a question.

You can use the West book. You can't use any other books or notes.

Unless stated otherwise, you can use as fact any theorem which I presented during the lectures. You can also use any theorems from the West book.

These problems all have reasonably short solutions. A very long solution will not get full marks, even if it is correct. The grader will subjectively decide what constitutes "very long".

All graphs are simple.

1. **(5 pts)** G is a planar triangulation (i.e. a planar embedding of a graph where every face has size 3). Prove that the number of faces is even.

2. **(10 pts)** A graph is said to be *uniquely k-edge-colourable* if it is k-edge-colourable and all k-edge-colourings are equivalent under a permutation of the colours. In other words, every k-edge-colouring partitions the edges into the same k matchings.

   Prove that, for $k \geq 2$, if a k-regular graph is uniquely k-edge-colourable then it has a Hamiltonian cycle.

   This is a generalization of 7.2.14 on page 296.

3. **(10 pts)** G is a bipartite graph with bipartition (X, Y) where $|X| = |Y| = n$. For every $S \subseteq X$ except for $S = X$ and $S = \emptyset$, we have $|N(S)| > |S|$. Prove that every edge of G lies in a perfect matching.

   Remark: this is roughly the same problem as 3.1.21 on page 119.

4. **(20 pts)**

   (a) **(5 pts)** Prove that for $k \geq 3$, every k-regular graph with exactly $2k + 1$ vertices is 3-connected.

   (b) **(15 pts)** Prove that for $k \geq 2$ every k-regular graph with exactly $2k + 1$ vertices has a Hamiltonian cycle. For part (b), you can use the following lemma (which you don't need to prove):

   Lemma: Every 2-connected graph with $n \geq 3$ vertices and minimum degree δ has a cycle of length at least $\min\{n, 2\delta\}$.

   This is essentially the same as 7.2.40 on page 298

5. **(15 pts)** Prove that if G is triangle-free (i.e. contains no cycles of length 3) then $\chi(G) \leq 2\sqrt{n}$, where n is the number of vertices in G.

   This is 5.2.15 on page 216. West provides the following hint: Use large neighbourhoods as colour classes while there remain vertices of high degree; then apply Brooks' Theorem.

6. **(25 pts)** H is a tournament, and $x \in H$ is a vertex with maximum outdegree.

   (a) **(5 pts)** Prove that every vertex $u \in H$ can be reached from x using a directed path with at most 2 edges.

   (b) **(20 pts)** Prove that H has a spanning tree T rooted at x such that

   (i) The edges of T are directed away from the root;

   (ii) The height of T is at most 2;

   (iii) Every vertex other than x has outdegree at most 2 in T.

   Hint: Use either network flows or Hall's Theorem.

   This is Exercise 4.3.16 from page 190.

   See Definition 1.4.27 from page 62.