Write your solutions in the exam booklets provided. Put your name on every booklet.

Ask me if you don’t understand a question.

You can use the West book. You can’t use any other books or notes.

Unless stated otherwise, you can use as fact any theorem which I presented during the lectures. You can also use any theorems from the West book.

These problems all have reasonably short solutions. A very long solution will not get full marks, even if it is correct. The grader will subjectively decide what constitutes “very long”.
All graphs are simple.

1. (5 pts) \( G \) is a planar triangulation (i.e. a planar embedding of a graph where every face has size 3). Prove that the number of faces is even.

2. (10 pts) A graph is said to be \textit{uniquely} \( k \)-edge-colourable if it is \( k \)-edge-colourable and all \( k \)-edge-colourings are equivalent under a permutation of the colours. In other words, every \( k \)-edge-colouring partitions the edges into the same \( k \) matchings.

Prove that, for \( k \geq 2 \), if a \( k \)-regular graph is uniquely \( k \)-edge-colourable then it has a Hamiltonian cycle.

This is a generalization of 7.2.14 on page 296.

3. (10 pts) \( G \) is a bipartite graph with bipartition \((X,Y)\) where \( |X| = |Y| = n \). For every \( S \subseteq X \) except for \( S = X \) and \( S = \emptyset \), we have \( |N(S)| > |S| \). Prove that every edge of \( G \) lies in a perfect matching.

Remark: this is roughly the same problem as 3.1.21 on page 119.

4. (20 pts)
   
   (a) (5 pts) Prove that for \( k \geq 3 \), every \( k \)-regular graph with exactly \( 2k + 1 \) vertices is 3-connected.
   
   (b) (15 pts) Prove that for \( k \geq 2 \) every \( k \)-regular graph with exactly \( 2k + 1 \) vertices has a Hamiltonian cycle. For part (b), you can use the following lemma (which you don’t need to prove):

   \textbf{Lemma:} Every 2-connected graph with \( n \geq 3 \) vertices and minimum degree \( \delta \) has a cycle of length at least \( \min\{n, 2\delta\} \).

   This is essentially the same as 7.2.40 on page 298.

5. (15 pts) Prove that if \( G \) is triangle-free (i.e. contains no cycles of length 3) then \( \chi(G) \leq 2\sqrt{n} \), where \( n \) is the number of vertices in \( G \).

   This is 5.2.15 on page 216. West provides the following hint: Use large neighbourhoods as colour classes while there remain vertices of high degree; then apply Brooks’ Theorem.

6. (25 pts) \( H \) is a tournament, and \( x \in H \) is a vertex with maximum outdegree.

   (a) (5 pts) Prove that every vertex \( u \in H \) can be reached from \( x \) using a directed path with at most 2 edges.

   (b) (20 pts) Prove that \( H \) has a spanning tree \( T \) rooted at \( x \) such that

   (i) The edges of \( T \) are directed away from the root;
   
   (ii) The height of \( T \) is at most 2;
   
   (iii) Every vertex other than \( x \) has outdegree at most 2 in \( T \).

   Hint: Use either network flows or Hall’s Theorem.

   This is Exercise 4.3.16 from page 190.

   See Definition 1.4.27 from page 62.