CSC 2410 Final Exam

Dec 15, 2006

Write your solutions in the exam booklets provided. Put your name on every booklet.

Ask me if you don't understand a question.

You can use the West book. You can't use any other books or notes.

You can use as fact any theorem which I presented during the lectures. You can also use any theorems from the West book.

These problems all have reasonably short solutions. A very long solution will not get full marks, even if it is correct. For example, a proof with many cases is too long. The grader will subjectively decide what constitutes "very long".

All graphs are simple, unless stated otherwise.

1. **(5 pts)** Prove that the 8-clique cannot be embedded on the torus.

   You can use Euler's formula for embeddings of non-planar graphs on the torus: $V - E + F = 0$.

2. **(10 pts)** G is any k-regular, $(k-1)$-connected graph with an even number of vertices. G' is obtained by removing any set of $k-1$ edges from G. Prove that G' has a perfect matching.

   Hint: this statement is a strengthening of Corollary 3.3.8 on page 139.

   Remark: this is a strengthening of Problem 3.3.16 on page 147.

3. **(10 pts)** For any $k \geq 3$, form G as follows: (1) Start with a k-clique; (2) delete an edge xy; (3) add a new vertex z which is adjacent only to x and to y. Thus G has $k+1$ vertices, one has degree 2, and all others have degree $k-1$. Prove that G has total chromatic number at least $k+1 = \Delta(G)+2$.

   Recall: a total colouring assigns colours to the vertices and edges so that adjacent vertices get different colours, edges that share an endpoint get different colours, and an edge has a different colour than both of its endpoints.

4. **(10 pts)** A *hyperedge* is a set of vertices. A *hypergraph* consists of a collection of vertices and a collection of hyperedges. So if every hyperedge contains exactly 2 vertices, then this is simply a graph.

   (a) **(6 pts)** Let G be a hypergraph with fewer than 2^{k-1} hyperedges, where every hyperedge has size at least $k \geq 2$. Prove that the vertices of G can be 2-coloured so that every hyperedge has at least one vertex of each colour.

   (b) **(4 pts)** Let G be a hypergraph with *exactly* 2^{k-1} hyperedges, where every hyperedge has size at least k. Prove that the vertices of G can be 2-coloured so that every hyperedge has at least one vertex of each colour.

   Hint: use the probabilistic method.

   Note: Part (a) is Problem 8.5.13(a) on page 449.

5. **(10 pts)** Suppose that G is a connected graph that is not a tree and that G has girth at least 5. Prove that $\overline{G}$ is Hamiltonian. ($\overline{G}$ is the complement of G.)

   Hint: Recall the following Theorem presented in class:

   Suppose H is a graph on n vertices such that for every pair of non-adjacent vertices u, v, we have $\deg(u) + \deg(v) \geq n$. Then H is Hamiltonian.

   Note: This is a special case of problem 7.2.25 on page 297.

6. **(10 pts)** You are given a graph G and you want to find a hidden subgraph H. You have the following hints: for each $v \in G$ you have two values $a(v), b(v) \geq 0$. If $a(v)$ is high, then this is a hint that $v \in H$, while if $b(v)$ is high then this is a hint that $v \notin H$. It is possible for this hints to be contradictory, eg perhaps $a(v), b(v)$ are both high. For each edge uv there is a value $x(uv)$. If $x(uv)$ is high then this is a hint that u, v are either both in H or both not in H.

   Design a polytime algorithm to find a most likely subgraph H. I.e., determine a subgraph H which maximizes:

$$\sum_{v \in H} a(v) + \sum_{v \notin H} b(v) - \sum_{u \in H, v \notin H} x(uv).$$