CSC 2410 Assignment #3, Fall 2017

Due: Tuesday Dec 5 at 10:00 AM

You may consult the text. You may not consult any other books and materials.

You may consult with each other **only on problem 6.** For the other problems, you can only consult with the instructor and the TA for this course.

Of course, each problem requires a well-written proof. Proofs that are unnecessarily lengthy might not get full marks, even if they are correct. And they might not be read thoroughly by the grader.

1. **(5 pts)** Either find a planar embedding of the Petersen Graph or find a K_5- or $K_{3,3}$-subdivision in the Petersen Graph. (The Petersen Graph is on the cover of the text; see also Def 1.1.36.)

2. **(10 pts)** Prove that a planar embedded graph is bipartite iff every face has even size.

3. **(25 pts)** 6.2.9 from West.

4. **(35 pts)** In this problem, you may not apply the Four Colour Theorem.

   (a) Prove that if G is a connected planar graph such that (i) G has maximum degree at most 5, and (ii) G has at least one vertex of degree less than 5, then $\chi(G) \leq 4$.

   (b) Prove that if G is a 3-connected 5-regular planar graph then $\chi(G) \leq 4$.

   Hint: Choose two non-adjacent vertices a, b and contract them into a single vertex. If you choose a, b wisely, then the resultant graph will be planar, and you will be able to prove that it cannot contain a 5-critical subgraph. Then argue that this implies $\chi(G) < 5$.

   (c) Prove that if G is a planar graph with maximum degree at most 5 then $\chi(G) \leq 4$.

   Note: If you do not solve (a) or (b), then you can assume that what they assert is true and still attempt (c).

5. **(15 pts)** You are given a multigraph with a label from $\{1, ..., k\}$ on each edge; there are no loops but there may be multiple edges. Your goal is to assign a label from $\{1, ..., k\}$ to each vertex so that there is no edge uv where u, v and uv all have the same label. This is a variation of k-colouring where each edge only forbids one colour from appearing on both endpoints; if uv is labelled 1 then u and v can have the same colour as long as it is not 1.

   Use the probabilistic method to prove that if the number of edges is at most k^2 then there is a way to label the vertices as required.

6. **(20 pts)** Suppose that G has n vertices and minimum degree δ. Suppose further that for every two vertices x, y in G, $|N(x) \cup N(y)| + \delta \geq n + 10$. Prove that G has a Hamilton cycle.

   Remark: The "+10" term is much higher than necessary.