CSC 2410 Spring 2009, Assignment #2

Due: Monday March 9.

You may consult the text. You may not consult any other materials. You may consult with each other **only on problems 1 and 2.** For the other problems, you can only consult with the instructor and the TA for this course.

Of course, each problem requires a well-written proof. Proofs that are unnecessarily lengthy might not get full marks, even if they are correct. And they might not be read thoroughly by the grader.

1. (10 pts) Let G be any 3-connected graph, and let e_1, e_2, e_3 be 3 edges of G such that (i) no two have a common endpoint (i.e. they form a matching of size 3) and (ii) $G - \{e_1, e_2, e_3\}$ is connected (i.e. the three edges don't form an edge cutset). Prove that G has a cycle which contains e_1, e_2, e_3. Describe how to find that cycle in polytime.

2. (10 pts) Let H be any simple bipartite graph where each side of the bipartition has size n, and where H has maximum degree at most $t \le \frac{n}{10}$. Show that there exists a $2t$-regular simple bipartite graph H' on the same vertex set, such that H is a subgraph of H'. Describe how you could find such a graph H' in polytime.

   (Remark: the constant '10' is not best possible; the statement remains true for much larger upper bounds on t.)

3. (a) (2 pts) Prove that if $|G| = n$ and $\chi(G) = t$ then $\overline{G}$ does not have a matching of size greater than $n - t$.

   (b) (8 pts) Use part (a) to prove that for all k, there is no k-regular graph with chromatic number k on $2k - 2$ vertices.

4. (10 pts) Problem 5.2.13 of West. (Here, you will prove the existence of graphs with girth 6 and arbitrarily high chromatic number.)

5. (10 pts) Consider a sequence of integers $d_1 \ge d_2 \ge ... \ge d_n \ge 1$. Prove that this is the degree sequence of a simple graph that has a perfect matching iff

   (i) $d_1, ..., d_n$ is the degree sequence of a simple graph; and

   (ii) $d_1 - 1, ..., d_n - 1$ is the degree sequence of a simple graph.

6. (15 pts) Recall that a 4-critical graph must have minimum degree 3. Prove that a 4-critical graph cannot have an even cycle, C, such that every vertex of C has degree 3 and such that the vertices of C do not form a clique. Generalize this for k-critical graphs, $k \ge 4$, and find a polytime algorithm which tests whether or not a graph satisfies this condition. (Note: you might be able to find the algorithm without completing the other parts of the problem.)