Lecture 11, Nov 21

Linear and Integer Programming

Network Flows Applications

Project selection – sec. 7.11(book)

A Linear Program

- A "linear program" is an optimization problem consisting of:
  - an "objective function":

    $$c_1 x_1 + c_2 x_2 + ... + c_n x_n$$

    c_i constants, x_i variables (all over real numbers).
  - m linear "constraints": i.e., of the form

    a_{i,1} x_1 + a_{i,2} x_2 + ... + a_{i,n} x_n<= /= / >= b_i for i = 1, 2, ..., m.

Linear Programming: Goal

- The goal is to find real values of x_i's that maximize/minimize the objective function and satisfy all constraints.
- This problem is solvable in polynomial time.
- Integer programming: the same problem, but all the constraints and the variables must be integer.
- Is integer programming harder than LP?

Is integer programming harder than the general LP?

- Yes.
- IP is NP-hard.
- We'll see a few examples of NP-hard problems that can be solved with integer programming.

A political problem [CLRS] – cont'd

policy	urban	suburban	rural
build roads	-2	5	3
gun control	8	2	-5
farm subsidies	0	0	10
gasoline tax	10	0	-2

Example 1: a political problem [CLRS]

- Suppose that you are a politician trying to win an election. Your district has three different types of areas—urban, suburban, and rural. These areas have, respectively, 100,000, 200,000, and 50,000 registered voters. To govern effectively, you would like to win a majority of the votes in each of the three regions. You are honorable and would never consider supporting policies in which you do not believe. You realize, however, that certain issues may be more effective in winning votes in certain places. Your primary issues are building more roads, gun control, farm subsidies, and a gasoline tax dedicated to improved public transit.

A political problem [CLRS] – cont'd

- Not hard to find a solution by trial and error.
- E.g. $20,000 to building roads, $0 for gun control, $4,000 to farm subsidies and $9,000 to gas tax.
- This is called a *feasible solution.*
- A total of $33,000 is spent.
- Is it optimal?

A political problem [CLRS] – cont'd

- According to your campaign staff's research, you can estimate how many votes you win or lose from each population segment by spending $1,000 on advertising on each issue. This information appears in the table of Figure 1. In this table, each entry describes the number of thousands of either urban, suburban, or rural voters who could be won over by spending $1,000 on advertising in support of a particular issue. Negative entries denote votes that would be lost. Your task is to figure out the minimum amount of money that you need to spend in order to win 50,000 urban votes, 100,000 suburban votes, and 25,000 rural votes.

An LP solution

- We formulate the problem in terms of linear programming.
- Denote by x_1 the number of thousands spent on advertising on building roads.
- x_2 the number of thousands spent on gun control.
- x_3 the number of thousands spent on farm subsidies.
- x_4 the number of thousands spent on gas tax.

The constraints

- At least 50,000 urban votes:

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$$

- At least 100,000 suburban votes:

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$$

- At least 25,000 rural votes:

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$$

No negative-cost advertising

- We are not allowed negative advertisement. Moreover, even negative advertisement costs money.

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ \text{and}\ x_4 \geq 0$$

- Want to minimize the total cost:

$$x_1 + x_2 + x_3 + x_4$$

Final LP formulation:

(29.6) minimize $x_1 + x_2 + x_3 + x_4$

subject to

$$\begin{aligned}
&(29.7)\ -2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50\\
&(29.8)\ 5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100\\
&(29.9)\ 3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25\\
&(29.10)\ x_1, x_2, x_3, x_4 \geq 0\ .
\end{aligned}$$

- The solution of this linear program will yield an optimal strategy for the politician.