A Brownian Motion Model for Last Encounter Routing

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***Abstract*— We use a mathematical model based on Brownian motion to analyze the performance of Last Encounter Routing (LER), a routing protocol for ad hoc networks. Our results show that, under our model, LER outperforms the simple flooding mechanism employed by reactive protocols.**

I. Introduction

Mobile ad hoc networks have attracted the attention of the networking community because of their decentralized and dynamic nature that may give rise to exciting new applications. However, the non-hierarchical and time-variant nature of ad hoc networks has also led to a multitude of design challenges. Many of these challenges are related to the difficulty of obtaining the global state of an ad hoc network. Information such as network topology, the positions of nodes or the data they carry may vary in time and is immediately accessible only on a local scale. Due to the absence of a central authority or an underlying infrastructure, approaches used in static networks that rely on the assessment of the global state are inappropriate for addressing problems that arise in this area. One of the most persistent of such problems is routing, which, in addition to being performed in a distributed manner, should be able to cope with frequent and abrupt changes in network topology due to mobility.

Traditional approaches in ad hoc routing include *proactive* or *table-driven* protocols and *reactive* or *demand-driven* protocols. In proactive protocols (such as DSDV [1] and WRP [2]), nodes maintain routing tables with topology information. Routing tables are updated through the frequent dissemination of update packets. Establishing a route to a destination is therefore straightforward. However, proactive protocols suffer from prohibitive maintenance overhead when nodes are highly mobile. In response to this behavior, reactive protocols that do not rely on routing tables (such as AODV [3] and DSR [4]) were proposed. In such protocols, nodes dynamically discover routes by flooding the network with route request packets. Although such protocols outperform proactive protocols under high mobility (see [5], [6]), they still generate a significant overhead while flooding during the route discovery phase.

There are several protocols in literature ([7]–[11]) that seem to belong in a gray area between reactive and proactive protocols. The general scheme that they conform to can be described as follows. Although nodes maintain routing tables as in proactive protocols, the information stored in them is not updated as often. As expected, this results in inconsistency since entries in the routing tables become inaccurate. Nonetheless, in order to route a packet, this information is used as if it were accurate and the packet is forwarded accordingly. Due to the aforementioned inaccuracy, this may lead to a routing failure. Upon such a failure, the protocols we describe resort to flooding, just like reactive protocols. However, instead of flooding to find the destination, as in the route discovery phase of reactive protocols, flooding is performed with the intention to locate nodes with more accurate routing information. This information is then used to initiate a new forwarding attempt. This process, alternating between forwarding and flooding, is repeated until the destination is reached.

We refer to these protocols as *approximate information protocols*, due to their inherent use of inaccuracy. An implied assumption here is that one can quantify the accuracy of information stored in routing tables. This is necessary, since when the protocol enters a flooding phase, it should be able to determine whether a node found through flooding has higher or lower accuracy than the one currently used. In that sense, a *measure of accuracy* of this information should exist.

An example of a protocol that conforms with this general scheme is Last Encounter Routing (LER), originally proposed by Grossglauser and Vetterli [7]. In LER, each node maintains in a table the position of each destination at some time in the past, in the spirit of position-based routing protocols. This serves as an estimate of where the destination currently is. The age of this estimate is also stored and serves as a measure of the estimate's accuracy. A source can route a packet based on this position information using any known forwarding strategy; this constitutes the forwarding phase. If the destination is not reached thus, *i.e.* a forwarding failure occurs, LER resorts to flooding. A node whose estimate of the destination's position is better than the current one is sought out. As the general scheme suggests, this better estimate is then used for a new forwarding attempt and the process is repeated until the destination is reached.

LER owes its name to the frugal update mechanism it employs: nodes update their entries every time a destination is within their transmission radius, *i.e.* when two nodes *encounter* each other while moving. This has two important implications with respect to the protocol's behavior. First, the maintenance overhead is negligible. Second, network mobility is solely responsible for the dissemination of routing information.

LER is quite simple, in the sense that it merely outlines a route discovery phase and does not explicitly address many of the issues that arise in real life scenarios, such as packet loss, route maintenance, loop freedom e.t.c. However, we believe it is worth investigating when viewed in the context of approximate information protocols. As we discussed, approximate information protocols route packets based on irregularly maintained information. A question that thus naturally arises is the following: When routing a packet, should one exploit inaccurate routing information by using an approximate information protocol, or should one avoid maintaining such information altogether and resort to simple flooding, like the one employed by reactive protocols?

In this paper, we attempt to answer this question by analyzing LER. Our goal is thus to discover whether LER behaves better than simple flooding. Compared to other approximate information protocols, LER is good starting point for such an investigation due to its simplicity. Furthermore, there are analytical [7] as well as simulation [7], [12] results in literature, which support that LER indeed behaves well.

Our main result, presented in Theorem 1, is that, assuming that nodes in the network perform Brownian motions, the answer to the above question is affirmative: LER is asymptotically better than simple flooding by a significant factor (see Section V). Though our result is restricted to LER under a specific mobility model, we believe that it can be seen as an indication that approximate information protocols in general are worth further investigation.

II. The LER Protocol

We will discuss two variations of the LER protocol in this paper, which we call LER_0 and LER_1. LER_0 is closer to EASE, the original version proposed by Grossglauser and Vetterli [7], though the two are not identical. The algorithm for LER_0 can be found in Fig. 1. Each node i maintains a routing table RT_i and an elapsed time table T_i. For any node j, RT_{ij} contains the coordinates of j at the time of the last encounter between i and j, and T_{ij} is the elapsed time since that encounter. More formally, let $X_j(t)$ denote the position of node j at time t and suppose that nodes i and j meet at time t_1 and then do not meet again up to and including time t_2. Then, at time t_2 the routing table of i will contain the entry $RT_{ij} = X_j(t_1)$ and the elapsed time table will contain the entry $T_{ij} = t_2 - t_1$. Throughout this paper, we refer to RT_{ij} as the i's *estimate* of node j's current position and to T_{ij} as the *accuracy* of this estimate. By definition, each node knows its own location with perfect accuracy, i.e. $RT_{ii} = X_i(t)$ and $T_{ii} = 0$ at any time t and for all i.

Routing happens in the above setting as follows. Suppose that a source s wishes to send a packet to a destination d at time T. Initially, the source node s can send it to the position $X_d(T - T_{sd})$, using any of the position-based forwarding strategies that exist in literature. If the destination is not reached with this forwarding (e.g. because it is not there anymore) the protocol resorts to flooding. While flooding, the protocol looks for any node n such that $T_{nd} < T_{sd}$. After locating such a node n, the protocol resumes forwarding using the position information $X_d(T - T_{nd})$. This process is repeated and the protocol alternates between forwarding and flooding until the destination is reached. An illustration of this behavior can be seen in Fig. 2.

Route a packet p from s to d at time T.
{
    $i = s$; forward p towards d using $X_d(T - T_{id})$;
    let a be the node reached at the end
                of this forwarding;
    while $(a \neq d)$ {
        flood the network from a until a node n
            is reached such that $T_{nd} < T_{id}$;
        forward p towards d using $X_d(T - T_{nd})$;
        let a be the node reached at the end
                of this forwarding;
        $i = n$;
    }
}

Fig. 1. The LER_0 algorithm

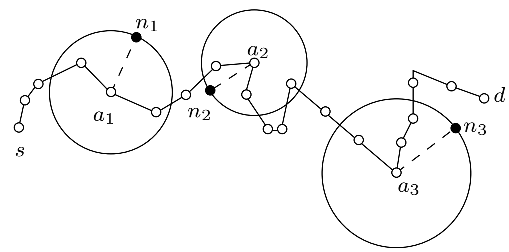

Fig. 2. An illustration of the execution of the LER algorithm.

We note that a node will always be located during the flooding phase, since at least one node with perfect accuracy exists, namely the destination. Grossglauser *et al.* use the term *anchor points* for the points a in which forwarding fails and flooding is initiated, and the term *messenger* nodes for the nodes n that satisfy $T_{nd} < T_{id}$, where T_{id} is the current accuracy. We also adopt the above terms throughout this paper, though we differentiate between messenger nodes and the destination (we will not refer to the destination as a messenger node, although its accuracy is less than the current one). Finally, we will refer to the new accuracy obtained through flooding, i.e. the accuracy of the messenger node closest to the anchor point, as the *improved* accuracy.

Both in the original paper on LER as well as in this one, flooding is assumed to expand in concentric circles around the current anchor point a until n, the closest messenger node (or the destination) is reached. The area flooded is thus defined as a circle centered at a with radius equal to the distance between a and n. In practice, this can be implemented by a TTL mechanism: areas of increasing size can be flooded sequentially, until a messenger node (or the destination) is reached. In fact, one can show that, if the area in the initial step is constant (i.e. it does not depend on the accuracy at the anchor point) and in each step the area flooded is doubled, the sum of all areas flooded in order to locate the closest messenger node will be at most four times the area that we consider here. Chang and Liu [13] have shown that the factor of 4 can be improved by using a randomized flooding strategy. In any case, the area we consider here is of the same order as the total area flooded by such an implementation.

On the other hand, when one floods a constant initial area, it is reasonable to use the estimate of the messenger node with the best accuracy within that area. This is not how LER_0, as defined above, would behave. Furthermore, as we will see in our analysis, the expected area that we flood in order to find the closest messenger node or the destination depends on the accuracy at the current anchor point. It therefore makes sense, instead of flooding a fixed initial area, to flood an area $A(T)$ whose size depends on the current accuracy T. In such an implementation, the sum of all areas flooded would be of the same order as the sum of $A(T)$ and the area we consider here.

LER_1, the second protocol we consider, aims at capturing this behavior. It is outlined in Fig. 3. In the flooding phase, an area A that depends on the present accuracy T_{id} is flooded. Of all the estimates of the destination's position from messenger nodes in A the most accurate one is used to forward the packet. However, it is possible that neither a messenger node nor the destination are within the area A. In this case the protocol floods again, this time behaving as LER_0.

Route a packet p from s to d at time T.
{
    $i = s$; forward p towards d using $X_d(T - T_{id})$;
    let a be the node reached at the end of this forwarding;
    while $(a \neq d)$ {
        flood an area $A(T_{id})$ around a.
        let M be the set of nodes k reached thus such that $T_{kd} < T_{id}$.
        if $M \neq \emptyset$ then let $n = \arg\min_{k \in M} T_{kd}$;
        else flood the network from a until a node n is reached such that $T_{nd} < T_{id}$;
        forward p towards d using $X_d(T - T_{nd})$;
        let a be the node reached at the end of this forwarding;
        $i = n$;
    }
}

Fig. 3. The LER_1 algorithm

III. Related Work

In their original paper [7], Grossglauser and Vetterli present an analysis for EASE, the variation of LER they originally proposed. Working on a mobility model similar to the one that we use here, they show that the total flooding cost of EASE is less than the cost of forwarding a packet over the shortest path to the destination, a result that they also support with simulations. Our result for LER_1, which we present in detail in Section V, is an improvement on their bound. Furthermore, their mathematical analysis is based on certain assumptions which do not appear in our present work. An analysis based on our model, under similar assumptions to the ones employed by Grossglauser and Vetterli yields stronger results than the ones we present in this paper. Such an analysis can be found in [14], which also investigates the optimality of the flooding condition as well as the behavior of the protocol with respect to the speed of the nodes and the network density. Interestingly, in spite of the simplifications used in [14], the cost obtained is roughly of the same order as the cost of LER_1 in the present paper.

Sarafijanovic-Djukic *et al.* [12] study EASE under the random waypoint mobility model. They introduce a modification of EASE that improves the estimate of the destination's position, given that the nodes move according to the random waypoint model, and demonstrate the relative improvement on protocol performance through simulations. These simulations also indicate that the cost is comparable to the source destination distance. FRESH, by Dubois Ferrière *et al.* [9], is a simpler version of LER. Its main difference is that nodes maintain only the accuracy tables T_i; the messenger's current position serves as an estimate for the destination's position. The authors use simulation results to argue that, as the protocol improves its accuracy, it also progresses in space and gets closer to the destination. GREP [8], [15] uses next-hop instead of position information and piggy-backing instead of last encounters as an update mechanism. To our knowledge, there is no performance analysis of GREP in existing literature. A proof of its loop freedom can be found in [15].

DREAM , by Basagni *et al.* [10], is a protocol that has some approximate information routing properties. Nodes maintain approximate position information and, in the context of the general scheme, route discovery consists only of a flooding phase and no forwarding phase; approximate information is only used to constrain flooding in the sense of directing it towards the destination. On the other hand, FSR, by Iwata *et al.* [11], has only a forwarding phase but no flooding phase. Using approximate next-hop information, it routes a packet accordingly and relies on an update mechanism that makes information at nodes closer to the destination more accurate.

Technically, LER is a *position-based* routing protocol (such as GPSR [16]), since the information used to route a packet to a destination is its position on the plane. Typically position-based protocols consist of a *forwarding strategy*, which specifies how a packet is routed given the destination's position, and a *location service*, which describes how the destination's position can be obtained. LER combines these two by introducing inaccuracy: each node has its own approximate location service, namely its routing table, whose information is gradually refined while routing.

IV. Model

Below we describe the mathematical model we use to analyze LER. Our network model consists of the distribution of nodes in the plane as well as the process that characterizes their mobility. Furthermore, we define a cost function, in order to evaluate protocol performance.

A. Network Model

Our network consists of an infinite, countable number of nodes. As in Section II, $X_i(t) \in \mathbb{R}^2$ will denote the position of node i at time t. Each one of these nodes moves independently according to a two-dimensional Brownian motion of variance proportional to a parameter v, which models the speed of node movement. This means that the displacement $X_i(t_2) - X_i(t_1)$ of a node i in a time interval $[t_1, t_2]$ is normally distributed with variance $v(t_2 - t_1)$ on each axis.

Furthermore, the set $\boldsymbol{\pi}(t) = \{X_1(t), X_2(t), \ldots\}$ of the positions of all nodes in the network is a Poisson field of density ρ spanning over the entire plane $\mathbb{R}^2$. In other words, at any given time nodes are uniformly distributed in $\mathbb{R}^2$ with density ρ. We note that there is no inconsistency in this assumption and the fact that nodes move: if at any time t_0 the network forms a Poisson field, and nodes move independently according to Brownian motions, the network also forms a Poisson field with density ρ at any $t \geq t_0$ -a proof of this can be found in Révész [17]. Route discoveries are considered to last for a negligible period of time, so that nodes are static while they take place. In addition, we assume that the node density ρ is large enough so that the network is connected and that a packet can always be forwarded successfully from one anchor point to another.

The destination is a node, not belonging to $\boldsymbol{\pi}$, that also performs a Brownian motion of variance proportional to v, independent of the rest of the nodes. We denote it with the index 0 and its trajectory with $X_0(t)$. An encounter between a node i and the destination occurs at time t if their distance is less than or equal to a transmission radius r_0, *i.e.* if $|X_i(t) - X_0(t)| \leq r_0$. To simplify notation, we assume that the length unit is normalized so that the transmission region around a node has unit area, i.e. $r_0 = 1/\sqrt{\pi}$. Under this assumption, our network model is fully characterized by two parameters: the variance v, which indicates the speed of nodes, and the density ρ. Finally, we assume that nodes have been moving in the time interval $(-\infty, 0]$ so that every node has encountered the destination with probability 1.

B. Cost Model

To discuss whether LER is better than simple flooding, one needs to introduce a cost as a performance metric under which the protocol's behavior can be evaluated. We would like to describe the expected total area flooded by the protocol from an anchor point that has accuracy T. However, computing this explicitly is quite difficult, so we actually define a cost that is slightly different.

Suppose that the destination is at the origin at time 0, and that at time T a flooding is initiated from the origin. We denote with $G(T)$ the expected area around the origin flooded in order to locate the destination or a messenger node, i.e. a node that met the destination in the time interval $(0, T]$. We will call $G(T)$ the *expected one-step flooding area*. Furthermore, we denote with $p(t, T)$ the density of the accuracy achieved though this flooding. In other words, $p(t, T)$ is the density of the accuracy of the node that we locate through this flooding. We will call $p(t, T)$ the *density of the improved accuracy*. Using this notation, we model the cost with a function Q that satisfies the following equation:

$$Q(T) = G(T) + \int_0^T Q(t)p(t, T)dt. \qquad (1)$$

This equation is an integral equation of a form known as a linear Volterra equation of the second kind (see *e.g.* Brunner et al. [18] for an exposition on the subject). From a probabilistic perspective, eq. (1) can be seen as the expected reward of a Markov chain with an uncountable number of states; $p(t, T)$ is then the conditional transition probability density and $G(T)$ the expected reward per transition. It is motivated by the fact that the total flooding area can be expressed as the area flooded at the first flooding step, plus the area flooded in all other steps. However, it differs from the expected total flooding area in the sense that flooding at subsequent steps should depend on flooding on previous ones, a behavior not captured by (1). As we will see though, this equation does capture many of the other features of the flooding phase of LER through the functions $G(T)$ and $p(t, T)$.

We use equation (1) to model the cost for both LER_0 and LER_1. More specifically, the cost of LER_0, denoted by $Q_0(T)$, will be defined by (1) with expected one-step flooding area $G_0(T)$ and density of the improved accuracy $p_0(t, T)$, whereas $Q_1(T)$, the cost of LER_1, will be similarly defined in terms of $G_1(T)$ and $p_1(t, T)$.

C. Discussion on the Model

In a real network, the movement of nodes is not described by Brownian motions. In that sense, assuming that nodes move according to Brownian motions is unrealistic. On the other hand, we believe that our analysis may be helpful in investigating different mobility models as well, including deterministic models. This is due to the fact that computing the cost of the protocol is reduced by our analysis to finding functions $G(T)$ and $p(t, T)$. Furthermore, there is intuition that suggests that the results for LER under different mobility models should not deviate considerably from the ones we observe here. We revisit these issues in more detail in our concluding remarks.

V. Overview of the Results

Our main result, presented in the following Theorem, is an asymptotic upper bound[1] for the cost of LER_1.

Theorem 1: For any $\alpha > 0$, there exists an initial area $A(T)$ such that the cost $Q_1(T)$ of protocol LER_1 is $o(\log^{2+\alpha} T)$.

This theorem states that, by choosing an appropriate $A(T)$, the cost Q_1 of LER_1 can be asymptotically upper-bounded by a polylogarithmic function of T. In fact, the exponent to which the logarithm is raised can be arbitrarily close to 2. Under our model, the cost of simple flooding (*i.e.* the cost of looking for the destination without using any approximate information) is

[1] Recall that $f = O(g)$ iff $\limsup_{t\to\infty} |f(t)|/g(t) \leq k$, for some $k \geq 0$, $f = o(g)$ iff $\lim_{t\to\infty} |f(t)|/g(t) = 0$ and $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

a linear function of T. Therefore, Theorem 1 implies that the cost of LER_1 is significantly lower than the cost of simple flooding under our model.

Before proving the above theorem, we first provide an analysis of LER_0 from which we obtain a much weaker result.

Theorem 2: The cost $Q_0(T)$ of protocol LER_0 is $O(T)$. Theorem 2 states that Q_0, the cost of LER_0, is asymptotically upper-bounded by a linear function. We thus merely prove that LER_0 is no worse than simple flooding.

Due to the relative simplicity of the LER_0 protocol compared to LER_1, many concepts that appear in the proofs of both of the above theorems are easier to introduce for LER_0. For this reason, in spite of the weak result it yields, we present the analysis of LER_0 first and then proceed with the analysis of LER_1.

VI. An Analysis of LER_0

One can compute the cost $Q_0(T)$ directly from functions $G_0(T)$ and $p_0(t,T)$ using equation (1). In the following two sections we describe these two quantities in terms of our model and then use them in the final section to derive an asymptotic upper bound for $Q_0(T)$.

A. *The expected one-step flooding area of* LER_0

To compute expected one-step flooding area $G_0(T)$, one has to describe the process that generates messenger nodes in our model. In particular, one needs to know how many messenger nodes exist at the time of a flooding and where these nodes and the destination are positioned on $\mathbb{R}^2$. Since we are interested in an asymptotic analysis, it suffices to find an asymptotic upper bound for $G_0(T)$ instead of an exact description.

To make these notions more precise we first give a formal definition of $G_0(T)$ in terms of our model. Suppose that at time 0 the destination was at the origin (*i.e.* $X_0(0) = 0$) and that at time T a flooding is initiated from $X_0(0)$. The set of messenger nodes is then defined as

$$C(T) = \{i : \exists t \in (0,T] \text{ s.t. } |X_i(t) - X_0(t)| \leq r_0\}. \quad (2)$$

We can then define $G_0(T)$ as follows

$$G_0(T) = \mathrm{E}\left[\min_{i \in C(T)\cup\{0\}} \pi |X_i(T)|^2\right]. \quad (3)$$

To compute this quantity, one has to describe the size of the set $C(T)$ as well as the positions $X_i(T)$, $i \geq 0$. All of the above are dependent random variables under our model. Before attempting to obtain a bound for $G_0(T)$, we discuss certain properties of these random variables in detail.

1) The number of nodes the destination encounters: We first try to describe the number of nodes that the destination meets in the interval $[0,T]$. This set differs from the set of messenger nodes because it includes the nodes that met the destination at time 0 but not at $(0,T]$. Before presenting this process for a destination that moves according to a Brownian motion, as it does in our model, we first consider the case where the destination follows a deterministic path. Suppose that the destination's trajectory is a continuous function $f : [0,+\infty) \to \mathbb{R}^2$, *i.e.* $X_0(t) = f(t)$, $t \geq 0$. The number of nodes the destination meets in $[0,t]$ is:

$$N_f(t) = |\{i : \exists s \in [0,t] \text{ s.t. } |X_i(s) - f(s)| \leq r_0\}|. \quad (4)$$

Then the following lemma holds.

Lemma 1: Let $N_f(t)$ be the number of nodes the destination meets in the interval $[0,t]$ given that it follows the fixed deterministic path $f(t)$. Then

$$\mathbf{P}\{N_f(t) = k\} = \frac{(\mu_f(t))^k}{k!} e^{-\mu_f(t)} \qquad t \geq 0, k \geq 0, \quad (5)$$

where $\mu_f(t) = \mathrm{E}[N_f(t)]$ is an increasing function and $\mu_f(0) = \rho$. Furthermore, $N_f(t)$ satisfies the independent increment property.

This is a generalization of a result given by Révész [17] for the case where $f(t)$ is a constant function (i.e. the destination is static). A proof of the general case can be found in [19]. This lemma indicates that $\{N_f(t), t \geq 0\}$ is a non-homogeneous Poisson process in which bulk arrivals can happen at time $t = 0$. The bulk arrivals at 0 are due to the fact that the number of nodes in the unit-area disk around the destination at time 0 is positive with non-zero probability. In fact, $N_f(0)$ is Poisson distributed with density ρ, by the nature of the Poisson field.

The expected values μ_f cannot be described with known functions in most cases. However, Révész cites the following result from Spitzer [20] regarding the asymptotic behavior of the expectation of number of nodes if f is constant, which we denote with $N_0(t)$:

$$\mu_0(t) = \mathrm{E}[N_0(t)] = \rho\left(\frac{2\pi vt}{\log vt} + (c_1 + o(1))\frac{vt}{\log^2 vt}\right) \quad (6)$$

where c_1 is a constant, independent of v,t and ρ. A very useful fact about $\mu_0(t)$, due to Quastel [21], is the following:

Lemma 2: Let f be any continuous function from $[0,+\infty)$ to $\mathbb{R}^2$. Then $\mu_0(t) \leq \mu_f(t)$.

A proof of this can be found in [19]. The above lemma states that any node that moves will meet, on average, more nodes than one that remains static.

Coming back to our model, in the case where the destination moves according to a Brownian motion with variance proportional to v (*i.e.* $X_0(t) = B(t), t \geq 0$), we can again define $N(t)$ similarly to eq. (4) as

$$N(t) = |\{i : \exists s \in [0,t] \text{ s.t. } |X_i(s) - B(s)| \leq r_0\}|. \quad (7)$$

The expectation of $N(t)$ can be described in terms of $\mu_0(t)$.

Lemma 3: Let $\mu(t) = \mathrm{E}[N(t)]$ be the expected number of nodes met by a destination that moves according to a Brownian motion in the interval $[0,t]$, and $\mu_0 = \mathrm{E}[N_0(t)]$ the expected number of nodes that a fixed destination meets in the same interval. Then $\mu(t) = \mu_0(2t)$.

A proof of this can be found in [14]. Contrary to the above expectation, the actual distribution of $N(t)$ is not as easily described explicitly: $N(t)$ can be expressed as an average of variables $N_f(t)$ over all possible paths f the destination may follow, and the average of Poisson random variables is, in general, not Poisson. Moreover, we note that $N(t)$ cannot be characterized by assuming that the destination is fixed and all other nodes move according to independent Brownian motions with variance $2v$; the distance between the destination and any point in the Poisson field is indeed such a Brownian motion, but these motions are not independent.

2) The spatial distribution of messenger nodes: As stated, apart from the number of nodes that the destination meets, $G_0(T)$ also depends on the positions of these nodes and the destination at time T. In particular, according to (3) one needs to know $X_i(T)$ for all $i \in C(T) \cup \{0\}$.

We define for each node i the epoch of the first encounter with the destination S_i as

$$S_i = \min\{t \geq 0 : |X_i(t) - X_0(t)| \leq r_0\}, \quad i \geq 1. \tag{8}$$

We note that if $\{N(t), t \geq 0\}$ is the counting process presented in the previous section, *i.e.* it is the number of messenger nodes encountered by the destination in the time interval $[0, t]$, then S_i, $i \geq 1$, coincide with the arrival epochs of this process (though they are not necessarily ordered the same way). Turning back our attention to (3), we try to describe the distributions of $X_i(T)$, $i \in C(T) \cup \{0\}$. The destination moves according to a Brownian motion in the interval $[0, T]$, and thus $X_0(T)$ is normally distributed around the origin with variance vT. The positions of the messenger nodes at time T can be written as

$$X_i(T) = X_0(S_i) + (X_i(S_i) - X_0(S_i)) + (X_i(T) - X_i(S_i)) \tag{9}$$

for $1 \leq i \leq |C(T)|$. The first summand is normally distributed with variance vS_i. The third is also normally distributed with variance $v(T - S_i)$. Finally, $|X_i(S_i) - X_0(S_i)|$ is upper-bounded by the transmission radius $r_0 = 1/\sqrt{\pi}$.

Equation (9) indicates that messenger nodes are normally distributed around the origin with variance vT, with an "error" displacement whose Euclidean length is bounded by a constant. In fact, when computing the asymptotic behavior of $G_0(T)$, one can safely ignore the effect of this displacement and consider messenger nodes normally distributed around the origin (see Lemma 10 in Appendix I).

Although the individual distribution of the position of each messenger node and the destination is easy to obtain, the same cannot be said for their joint distribution. To see this, note that these distributions are not independent; although the latter of the displacements eq. (9) comprises of are independent of each other and the destination's trajectory[2], the first ones are not. $X_i(T)$ are independent on the other hand given the trajectory of the destination, in which case however the first displacements are no longer normal -they are deterministic.

3) An asymptotic upper bound for $G_0(T)$: According to the above, while computing $G_0(T)$, one has to address two issues: first, the distribution of the number of messenger nodes is not easy to describe. Second, messenger nodes may be (almost) normally and identically distributed, but they are not independent.

Not knowing the distribution of the messenger nodes can be overcome by using Lemmas 1 and 2: one can compute G_0 by considering the distribution of the number given that the destination moves according to a fixed path, and then obtain an upper bound by taking the expectation over all paths. Lemma 2 makes this computation easier, as it lower-bounds μ_f for every f. For more details can be found in the proof of Lemma 4.

We work around the dependence between messenger node positions using the following idea: Suppose that, while flooding, instead of looking for the destination or any node that met the destination in the interval $(0, T]$, we only look for the destination or any node that met the destination in the interval $(0, \tau]$, where $\tau \leq T$. These nodes are a subset of the nodes that we would look for in our original setting, hence the area that we flood is an upper bound on $G_0(T)$. On the other hand, as seen in eq. (9), the dependent parts of the displacements are the ones up to the time of the first encounter. Therefore, if τ is very small, the nodes that met the destination in the interval $[0, \tau]$ can be considered "approximately" independent. This is formalized in the following lemma, whose proof is in Appendix I.

Lemma 4: If $\tau = \tau(T)$ such that $\tau(T) \leq T$ and $\lim_{T\to\infty} \tau(T)/T = 0$, then

$$G_0(T) = O\left(\frac{2\pi vT}{\mu_0(\tau(T)) - \mu_0(0)}\right)$$

where $\mu_0(t)$ is the expected number of nodes met by a static destination described by eq. (6).
By properly choosing $\tau(T)$ one can obtain an upper bound of $G_0(T)$ from Lemma 4. In fact, one can show that $G_0(T)$ is logarithmic in terms of T:

Lemma 5: $G_0(T) = O(\rho^{-1} \log vT)$.
The proof of this lemma can be found in Appendix II.

B. The density of the improved accuracy of LER$_0$

The density of the improved accuracy $p_0(t, T)$ depends not only on the number of messenger nodes and their positions on the plane, as $G_0(T)$ does, but also on the estimates these nodes have. This makes $p_0(t, T)$ harder to describe. Since we are interested in an upper bound, we do not need to calculate $p_0(t, T)$ explicitly: it suffices to find a density $\hat{p}_0$ that *stochastically dominates* p_0. *i.e.* one with the property

$$\int_t^T p_0(\tau, T) d\tau \leq \int_t^T \hat{p}_0(\tau, T) d\tau.$$

This means that an accuracy distributed according to $\hat{p}$ is more likely to be worse (take large values) than one following p_0. Roughly, this implies that more flooding steps would be necessary to reach the destination, a behavior that would increase the cost described by the corresponding Volterra equation.

One such accuracy, whose distribution stochastically dominates $p_0(t, T)$, is the largest accuracy in the interval $[0, T]$,

[2]We note that this is true because S_i is the first meeting times of node i with the destination, and node i may meet the destination again in the interval $[S_i, T]$. If S_i were *last* meeting times, this would not hold.

i.e. the accuracy of the node that met the destination at the earliest time possible. This yields the following result:

Lemma 6: For $0 < t \leq T$ and $\mu(t) = \mu_0(2t)$,

$$\int_t^T p_0(\tau, T) d\tau \leq 1 - e^{\mu(t)-\mu(T)}. \quad (10)$$

The proof of this statement can be found in Appendix III.

C. An asymptotic upper bound for the cost of LER$_0$

Lemmas 5 and 6 imply Theorem 2. One can compute an asymptotic upper bound for $Q_0(T)$ by solving a Volterra equation in which the functions $G(T)$, $p(t, T)$ are obtained by the above two lemmas. The proof can be found in Appendix IV.

We note that the bound in Theorem 2 indicates that the cost $Q_0(T)$ is asymptotically upper-bounded by the cost of simple flooding, which is linear. However, as observed in Section V, this fails to show that employing LER actually pays off, in the sense that our bound is at most linear but not better than linear. We believe however that this bound is not tight. We obtained it assuming that p_0 behaves as the distribution of the worse messenger; this suggests that we misestimated the improved accuracy at each step.

A more precise description of $p_0(t, T)$ should improve this bound. However, this would be quite involved, since it would entail understanding how the quantities we studied above (the number of messenger nodes, their positions and their accuracies) behave conditioned on the position of the closest messenger node to the origin. LER$_1$ however allows us to avoid such an analysis, as, under proper assumptions, the messenger node with the best accuracy in the initial area, instead of the closest one to the origin, will be chosen.

VII. An analysis of LER$_1$

Recall that protocol LER$_1$ differs from LER$_0$ in how flooding is conducted at each anchor point. Instead of looking for the closest messenger node, the protocol first floods an area $A(T)$ and, if any messenger nodes are in this area, it uses the most accurate estimate among the ones these nodes have. If, on the other hand, no messenger nodes are located thus, it floods again looking for the closest messenger node, as LER$_0$.

The process that generates messenger nodes is the same in both protocols. Therefore, the facts we presented in Sections VI-A.1 and VI-A.2 also apply for LER$_1$. However, since the process that defines which messenger node is picked is different, $G_1(T)$ and $p_1(t, T)$ are not the same as $G_0(T)$ and $p_0(t, T)$. In the following two sections, we give an asymptotic upper bound for $G_1(T)$ and, after deriving an asymptotic property of $p_1(t, T)$, we prove Theorem 1.

A. The expected one-step flooding area of LER$_1$

One can express $G_1(T)$, the expected one-step flooding area of LER$_1$, in terms of $A(T)$, the initial flooding area, and $G_0(T)$, the expected one-step flooding area of LER$_0$. The upper bound for $G_0(T)$ in Lemma 5 can be used to compute an upper bound for $G_1(T)$, giving the following result:

Lemma 7: $G_1(T) = O(A(T) + \rho^{-1} \log vT)$.

Proof: Let $\Gamma(T)$ be the area that we flood under protocol LER$_0$, i.e. $G_0(T) = \mathrm{E}[\Gamma(T)]$. Note that, if X is a positive random variable, then $\mathrm{E}[X] = \int_0^\infty \mathbf{P}\{X > y\} dy$ (see e.g. [22]). $G_1(T)$ can be written as

$$\begin{aligned} G_1(T) &= A(T) + \mathrm{E}[\Gamma(T) \mid \Gamma(T) > A(T)] \mathbf{P}\{\Gamma(T) > A(T)\} \\ &= A(T) + \dots \\ &\quad \int_0^\infty \mathbf{P}\{\Gamma(T) > y \mid \Gamma(T) > A(T)\} dy \mathbf{P}\{\Gamma(T) > A(T)\} \\ &= A(T) + \int_0^\infty \mathbf{P}\{\Gamma(T) > y \cap \Gamma(T) > A(T)\} dy \\ &= (1 + \mathbf{P}\{\Gamma(T) > A(T)\}) A(T) + \dots \\ &\quad \int_{A(T)}^\infty \mathbf{P}\{\Gamma(T) > y\} dy \leq 2A(T) + \mathrm{E}[\Gamma(T)] \end{aligned}$$

The above inequality along with Lemma 5 completes the proof. ■

B. An asymptotic upper bound for the cost of LER$_1$

The key idea behind the proof of Theorem 1 is the following. If the initial flooding area $A(T)$ is taken to grow faster than the expected area $G_0(T)$, which is $O(\log T)$ by Lemma 5, with high probability a messenger node will be within this area for large enough T. In fact, in the following lemma we prove that, if the area grows as $\Theta(\log^{1+2\alpha} T)$, $a > 0$, not only will this area include a messenger node, it will actually include one with an accuracy smaller than $T - \tau(T)$, where $\tau(T) = \beta \frac{T}{\log^\alpha T}$ and β any value in $(0, 1)$.

Lemma 8: Let $\alpha > 0$, $0 < \beta < 1$. Let $A(T) = \Theta\left(\log^{1+2\alpha} T\right)$ and define $\tau(T) = \beta \frac{T}{\log^\alpha T}$. Finally, let $H_{\tau(T)}$ be the event that a node with accuracy better than $T - \tau(T)$ is located by LER$_1$. Then

$$\lim_{T\to\infty} \mathbf{P}\left\{H_{\tau(T)}\right\} = \lim_{T\to\infty} \int_0^{T-\tau(T)} p_1(t, T) dt = 1$$

Proof: Define $\tau_1(T) = \tau(T) = \beta \frac{T}{\log^\alpha T}$ and $\tau_2(T) = \frac{T}{\log^\alpha T}$. Let $\Gamma_{\tau_1,\tau_2}(T)$ be the area one would need to flood (as in LER$_0$) in order to find the destination or a messenger node that met the destination in the interval $[\tau_1(T), \tau_2(T)]$. Then, by proceeding as in the proof of Lemma 4 one can show that

$$\mathrm{E}[\Gamma_{\tau_1,\tau_2}(T)] = O\left(\frac{2\pi vT}{\mu_0(\tau_2(T) - \tau_1(T))}\right) = O\left(\log^{1+\alpha} T\right)$$

Note that $\{\Gamma_{\tau_1,\tau_2}(T) \leq A(T)\}$ implies $H_{\tau(T)}$. By Markov's inequality

$$\mathbf{P}\{\Gamma_{\tau_1,\tau_2}(T) > A(T)\} \leq \frac{\mathrm{E}[\Gamma_{\tau_1,\tau_2}(T)]}{A(T)} = O(1/\log^\alpha(T))$$

and the Lemma follows. ■

Intuitively, for large values of T the improved accuracy will be smaller than $T - \tau(T)$ with high probability. This however allows us to derive an asymptotic upper bound on the number of times the protocol will resort to flooding: one can show that this will be of the order of $\log^{1+\alpha} T$. This observation gives us the following lemma, the proof of which can be found in Appendix V:

Lemma 9: Let $A(T) = \Theta\left(\log^{1+2\alpha} T\right)$, for some $\alpha > 0$. Then $Q_1(T) = O(\log^{2+3\alpha}(T))$.

Theorem 1 is an immediate corollary: to obtain the theorem for some $\alpha > 0$, apply Lemma 9 with $\alpha' = \frac{\alpha}{4}$.

We note that our choice of the function $\tau(T)$ was made in order to make the result of Theorem 1 concrete. In fact, by choosing $\tau(T)$ as $\frac{T}{g(T)}$, provided that $g(T)$ satisfies certain conditions, Theorem 1 can be extended to a bound of the form $o(g(T)\log^2 T)$ where $g(T)$ grows arbitrarily slow.

VIII. Conclusions

Theorems 2 and 1 suggest that LER outperforms simple flooding under a Brownian mobility model. It remains an open question whether the bound on the behavior of LER_0 can be improved. An analysis under a simplified model [14] suggested that the cost Q_0 may grow at most as $\log^2 T$, a bound that is slightly better than our bound for LER_1. However, Theorem 1 suggests that LER_1 provides a provably low cost that can be arbitrarily close to $\log^2 T$, which should be sufficient in practice. Therefore, if flooding is implemented with a TTL mechanism as described in Section II, and an initial area has to be flooded at each flooding step, one should choose this area as a function of the current accuracy, as LER_1 dictates.

An aspect not investigated was how the protocol's behavior is influenced by the network density ρ and the speed of the nodes, modeled here by parameter v. The simplified analysis of [14] indicated "nice scalability" properties of the protocol with respect to these parameters; in particular, the cost was asymptotically indifferent to the values of these parameters. Intuitively, LER exhibits an interesting counter-balance effect with respect to mobility, which explains the aforementioned invariance to node speed: a destination that moves fast or, more generally, its trajectory is described by a mobility process which is quite versatile, is likely to move far away from the anchor point and thus be harder to locate. On the other hand, a destination that moves fast should also encounter more nodes on the way, thus generating more messenger nodes and facilitating flooding, which in turn should decrease the cost. Similar intuition also exists with respect to node density. We note that the analysis of LER_1 could be extended towards showing such properties, though such an extension lies beyond the scope of this paper.

It is not easy to speculate whether analyses under different mobility models, such as the random waypoint model, would confirm the results that we see here. However, the counter-balance effect with respect to node speed mentioned above suggests that results under other mobility models may also confirm the good performance of LER. We note that, in the context of our approach, analyzing the protocol under different models reduces again to describing the expected one-step flooding area and the density of the improved accuracy, and solving the corresponding Volterra equation.

In LER, routing table updates happen through node encounters. However, it is not realistic to assume such encounters will happen among any two mobile nodes in a real-life network. An update mechanism that relies on transmission of routing information, such as by piggypacking a source's current location over data packets (in the spirit of GREP [8]) seems more appealing in a real-life scenario. It would thus be interesting to see whether the mathematical framework presented here can be used to analyze such a protocol.

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APPENDIX I
PROOF OF LEMMA 4

Let τ be a time such that $0 \leq \tau \leq T$. Then $C(\tau)$ is the set of messenger nodes that met the destination in the interval $(0, \tau]$. Let $C'(\tau)$ be the set of nodes that met the destination in the interval $(0, \tau]$ but not at time 0. Then $C'(\tau) \subseteq C(\tau) \subseteq C(T)$. We therefore have that $G(T)$ is less than or equal to

$$G_0^\tau(T) = \mathrm{E}\left[\min_{i \in C'(\tau) \cup \{0\}} \pi |X_i(T))|^2\right]. \tag{11}$$

Let S_i be the epoch of first encounter of node i with the destination, as defined in (8). Then, for all $i \in C'(\tau)$, we have $0 < S_i \leq \tau$. Furthermore, if $\{N(t), t \geq 0\}$ is the process defined in eq. (7), $|C'(\tau)| = \tilde{N}(\tau)$, where $\tilde{N}(t) = N(t) - N(0)$. W.l.o.g. we assume that nodes are indexed so that $C'(\tau) = \{1, 2, \ldots, \tilde{N}(T)\}$. Under this convention $G^\tau(T)$ can be written as:

$$G_0^\tau(T) = \mathrm{E}\left[\min_{0 \leq i \leq \tilde{N}(\tau)} \pi |X_i(T))|^2\right]. \tag{12}$$

By abusing notation, we define $S_0 = \tau$. We can show then that assuming that nodes are normally distributed around the origin (i.e. ignoring the bounded displacement in (9)) gives the same upper bound on the asymptotic behavior of $G_0^\tau(T)$. This is merely a simple consequence of the triangle inequality and the fact that $r_0 = 1/\sqrt{\pi}$:

Lemma 10: Let

$$\hat{G}_0^\tau(T) = \mathrm{E}\left[\min_{0 \leq i \leq \tilde{N}(\tau)} \pi |X_0(S_i) + (X_i(T) - X_i(S_i))|^2\right],$$

where S_i, $i \geq 1$ is defined by eq (8) and $S_0 = \tau$. Then $G_0^\tau(T) \leq \hat{G}_0^\tau(T) + 1$.

Since we are only interested in asymptotic behavior, we can focus on computing $\hat{G}_0^\tau(T)$. Consider $\hat{G}_0^{\tau|f}(T)$, the above quantity conditioned on the fact that the destination follows a deterministic path f in the interval $[0, \tau]$. Formally,

$$\hat{G}_0^{\tau|f}(T) = \mathrm{E}\left[\min_{0 \leq i \leq \tilde{N}_f(\tau)} \pi |f(S_i) + (X_i(T) - X_i(S_i))|^2\right].$$

where $\tilde{N}_f(\tau) = N_f(\tau) - N_f(0)$, and $N_f(t)$ is the number of nodes met by the destination described by (5) in Lemma 1. We note that $\tilde{N}_f$ is Poisson distributed with expectation $\tilde{\mu}_f(\tau) = \mu_f(\tau) - \mu_f(0)$. Then, $\hat{G}_0^\tau(T)$ is the expectation of $\hat{G}_0^{\tau|f}(T)$ over all possible paths the destination may follow in the interval $[0, \tau]$. The quantities $X_i(T) - X_i(S_i)$ are normal, independent random variables with variances $v(T - S_i)$, whereas $|f(S_i)|$ are upper-bounded by $\alpha_f = \max_{t \in [0,\tau]} |f(t)|$, *i.e.* the maximum Euclidean distance of the destination from the origin in the interval $[0, \tau]$.

We further condition $\hat{G}_0^{\tau|f}(T)$ on the number of nodes met by the destination in $[0, \tau]$:

$$\hat{G}_0^{\tau|f,n}(T) = \mathrm{E}\left[\min_{0 \leq i \leq n} \pi |f(S_i) + (X_i(T) - X_i(S_i))|^2\right].$$

$\hat{G}_0^{\tau|f}(T)$ can then be written as

$$\hat{G}_0^{\tau|f}(T) = \sum_{n=0}^{\infty} \hat{G}_0^{\tau|f,n}(T) \mathbf{P}\left\{\tilde{N}_f(\tau) = n\right\} \tag{13}$$

Let $\Gamma(T) = \min_{0 \leq i \leq n} \pi |f(S_i) + (X_i(T) - X_i(S_i))|^2$. Then

$$\hat{G}_0^{\tau|f,n}(T) = \int_0^\infty \mathbf{P}\{\Gamma(T) > y\} \, dy \tag{14}$$

It thus suffices to upper-bound $\mathbf{P}\{\Gamma(T) > y\}$. We have that

$$\mathbf{P}\left\{\Gamma(T) > \pi r^2\right\} = \mathbf{P}\left\{\min_{0 \leq i \leq n} |X_i(T) - X_i(S_i) + f(S_i)| > r\right\}$$

By independence, the r.h.s. is equal to the product of $\mathbf{P}\{|X_i(T) - X_i(S_i) + f(S_i)| > r\}$ for $0 \leq i \leq n$. This is equal to

$$\prod_{i=0}^{n} \mathbf{P}\{|(X_i(T) - X_i(S_i))/\sigma_i + f(S_i)/\sigma_i| > r/\sigma_i\}$$

where $\sigma_i^2 = v(T - S_i)$ the variances of $X_i(T) - X_i(S_i)$. Then, $(X_i(T) - X_i(S_i))/\sigma_i$ are normal with variance 1, We will make use of the following fact:

Lemma 11: Let $X = (X_1, X_2)$ where X_1, X_2 are independent, normal, zero-mean, one-dimensional random variables with variance 1. Let $\alpha \in \mathbb{R}^2$, $|\alpha| \geq 0$. Then there exists a random vector $Y = (Y_1, Y_2)$ where Y_1,Y_2 are independent, normal, zero-mean, one-dimensional random variables with variance $2\exp(2|\alpha|^2)$ s.t. $\mathbf{P}\{|X + \alpha| > r\} \leq \mathbf{P}\{|Y| > r\}$.

The above states that the Euclidean length of a normal random variable with variance 1 centered at a is stochastically dominated by the length of a normal random variance centered at the origin with a variance that depends on $|a|$. It can be proved from basic principles of normal random variables. Hence, by Lemma 11 we get that

$$\mathbf{P}\left\{\Gamma(T) > \pi r^2\right\} \leq \prod_{i=0}^{n} \mathbf{P}\{|Y_i(|\alpha_i|/\sigma_i)| > r/\sigma_i\}$$

where $Y_i(x)$ are independent normal with variance $\sigma^2(x) = 2\exp(2x^2)$. Since $v(T - \tau) \leq \sigma_i^2 \leq vT$ and $|\alpha_i| \leq \alpha_f$, the r.h.s is upper-bounded by

$$\prod_{i=0}^{n} \mathbf{P}\left\{|Y_i(\alpha_f/\sqrt{v(T-\tau)})| > r/\sqrt{vT}\right\}$$

The last quantity can be computed explicitly for normal densities and gives us

$$\mathbf{P}\{\Gamma(T) > y\} \leq \exp\left(-\frac{(n+1)y}{2\pi vT\sigma^2(\frac{\alpha_f}{\sqrt{v(T-\tau)}})}\right). \tag{15}$$

In other words, $\Gamma(T)$ is stochastically dominated by an exponentially distributed random variable and therefore by (14)

$$\hat{G}_0^{\tau|f,n}(T) \leq \frac{2\pi vT}{n+1}\sigma^2(\alpha_f/\sqrt{v(T-\tau)}).$$

Using the above bound, eq. (13) and the fact that, by Lemma 1, $\tilde{N}_f(\tau)$ is Poisson distributed with expectation $\tilde{\mu}_f(\tau)$ we get that

$$\hat{G}_0^{\tau|f}(T) \leq \frac{2\pi vT}{\tilde{\mu}_f(\tau)}\sigma^2\left(\alpha_f/\sqrt{v(T-\tau)}\right).$$

By Lemma 2 however, $\tilde{\mu}_f(\tau) \geq \tilde{\mu}_0(\tau)$, as $\mu_f(0) = \mu_0(0) = \rho$. By first using this to strengthen the above inequality and then taking the expectation over all f we get

$$\hat{G}_0^{\tau}(T) \leq \frac{2\pi vT}{\tilde{\mu}_0(\tau)}\mathrm{E}\left[\sigma^2\left(\alpha/\sqrt{v(T-\tau)}\right)\right]$$

where $\alpha = \max_{t\in[0,\tau]}|B(t)|$ is the maximum distance from the origin in $[0,\tau]$ of a two-dimensional Brownian motion of variance proportional to v. Suppose now that $\tau = \tau(T)$ is such that the limit of $\tau(T)/T$ as T goes to infinity is 0. We remind the reader that $\sigma^2(x) = 2\exp 2x^2$. We state the following Lemma without proof, for reasons of brevity.

Lemma 12: Let α_n be the maximum value of the radial part of a standard two-dimensional Brownian motion on an interval $[0,\xi_n]$, i.e. $a_n = \max_{0\leq t\leq\xi_n}\left((B_1(t))^2+(B_2(t))^2\right)^{1/2}$ where $B_1(t)$ and $B_2(t)$ are independent one-dimensional standard Brownian motions and $\xi_n \geq 0$ with $\lim_{n\to\infty}\xi_n = 0$. Then, for $c > 0$ and for large values of n, $\mathrm{E}\left[\exp ca_n^2\right] \leq \frac{4}{1-2c\xi_n}$, *i.e.* $\mathrm{E}\left[\exp ca_n^2\right]$ is upper-bounded by a constant for large n.

We note that a Brownian motion of variance proportional to v in the interval $[0,T]$ can be transformed to a standard Brownian motion in $[0,1]$ if it is divided by $\sqrt{vT}$. This fact along with Lemma 12 above complete the proof. ■

Appendix II
Proof of Lemma 5

Before proving Lemma 5, we first give an intermediate result, which is an implication of Lemma 4.

Lemma 13: Let $g : [0,\infty) \to \mathbb{R}$ be such that $g(t) = o(t)$ and $\lim_{t\to\infty} g(t) = +\infty$. Then, the expected one-step flooding area $G_0(T)$ is $O(g(T)\rho^{-1}\log vT)$.

Proof: Since $g(t) = o(t)$ and $\lim_{t\to\infty} g(t) = +\infty$, there exists a t_0 such that, for all $t > t_0$, $g(t) > 1$ and $g(t) < vt$. Let $\tilde{\mu}(t) = \mu_0(t) - \mu_0(0)$. We define $\tau(T)$ as

$$\tau(t) = \begin{cases} t, & t \leq t_0 \\ \tilde{\mu}^{-1}\left(\frac{\tilde{\mu}(t)}{g(t)}\right), & \text{o.w.} \end{cases}$$

Note that $\tau(t) \leq t$ for all $t \geq 0$, by the monotonicity of μ_0 (it is increasing by Lemma 1) and the fact that $g(t) > 1$ for $t \geq t_0$. We claim that, for $t > t_0$, the following inequality holds:

$$\frac{\tilde{\mu}(t)}{g(t)} \leq \tilde{\mu}\left(\frac{t}{g(t)}\right). \tag{16}$$

From (6) we have that $\tilde{\mu}(t) = \rho\left(\frac{2\pi vt}{\log vt} + (c_1+o(1))\frac{vt}{\log^2 vt}\right)$ as $-\mu_0(0)$ gets absorbed in the $o(1)$ term. The above inequality can thus be written as

$$\frac{2\pi}{\log vt} + (c_1+o(1))\frac{1}{\log^2 vt} \leq \frac{2\pi}{\log\frac{vt}{g(t)}} + (c_1+o(1))\frac{1}{\log^2\frac{vt}{g(t)}}$$

It suffices thus that $1/\log vt \leq 1/\log(vt/g(t))$, which is true since $vt > g(t)$ and $g(t) > 1$. We therefore get that

$$\lim_{t\to\infty}\frac{\tau(t)}{t} = \lim_{t\to\infty}\frac{\left(\tilde{\mu}^{-1}\left(\frac{\tilde{\mu}(t)}{g(t)}\right)\right)}{t} \leq \lim_{t\to\infty}\frac{\left(\tilde{\mu}^{-1}\left(\tilde{\mu}\left(\frac{t}{g(t)}\right)\right)\right)}{t}$$

from inequality (16) and the monotonicity of $\tilde{\mu}^{-1}$. The above however is equal to $\lim_{t\to\infty} 1/g(t)$ which is zero by our second assumption on $g(t)$. Hence, the conditions of Lemma 4 apply for $\tau(T)$ and $G_0(T) = O\left(\frac{2\pi vT}{\tilde{\mu}_0(\tau(T))}\right)$. For $T \geq t_0$,

$$\frac{2\pi vT}{\tilde{\mu}(\tau(T))} = \frac{2\pi vT}{\tilde{\mu}\left(\tilde{\mu}^{-1}\left(\frac{\tilde{\mu}(T)}{g(T)}\right)\right)} = \frac{2\pi vT}{\frac{\tilde{\mu}(T)}{g(T)}} = \frac{g(T)}{\rho}\log vT\cdot O(1)$$

and therefore $G_0(T) = O(\frac{g(T)}{\rho}\log vT)$. ■

Note that g in Lemma 13 can grow arbitrarily slow - provided that it goes to infinity. Showing that Lemma 5 holds thus requires to extend Lemma 13 to the case where g is a bounded function. Let $f(T)$ be

$$f(T) = \begin{cases} 0, & vT \leq 2 \\ G_0(T)/\log vT, & o.w. \end{cases}$$

To show that $G_0(T)$ is $O(\log(vT))$, it suffices to show that there exist a $T_0 > 0$ and an $M > 0$ such that, for all $T > T_0$, $f(T) \leq M$. We will prove this by contradiction.

Suppose that for all $T_0 > 0$, $M > 0$ there exists a $T > T_0$ such that $f(T) > M$. We can define the following sequence $\{t_i, i \geq 1\}$: Let $t_1 = 1$, and t_{i+1} be such that $t_{i+1} > \max(t_i, i)$ and $f(t_{i+1}) > \max(f(t_i), i)$. Our hypothesis implies that such a sequence exists (take $T_0 = \max(t_i, i)$ and $M = \max(f(t_i), i)$). By construction, the sequence $f(t_i)$ is increasing and, since $f(t_{i+1}) > i$, its limit is $+\infty$. For the same reasons t_i is also increasing and unbounded. Furthermore, note that, by Lemma 13, $G_0(T) = O(\log T\log vT)$, hence the limit of $f(t_i)/t_i \leq K\log t_i/t_i$ as i increases is 0. Hence the sequence $f(t_i)$ satisfies the conditions of Lemma 13, and so does $\sqrt{f(t_i)}$. Thus, for a large enough i,

$$G_0(t_i) \leq c\sqrt{f(t_i)}\log vt_i. \tag{17}$$

On the other hand, as $i \to \infty$ we have $\sqrt{f(t_i)}/f(t_i) \to 0$, hence for large enough i it is true that $\sqrt{f(t_i)} < \frac{1}{c}f(t_i)$ where c the constant in (17). Hence, for large enough i, we have that $G_0(t_i) < f(t_i)\log vt_i = G_0(t_i)$, a contradiction. ■

Appendix III
Proof of Lemma 6

We first define $p_0(t,T)$ formally. Suppose that at time 0 the destination was located at the origin and that at time T

a flooding is initiated at the origin. For each node $i \geq 1$ the elapsed time since the last encounter with the destination is

$$S_i^* = \min\{t \geq 0 : |X_i(T-t) - X_0(T-t)| \leq r_0\}. \quad (18)$$

W.l.o.g. we assume this time that nodes are indexed according to their accuracies. Under this convention, $S_1^* \leq S_2^* \leq \dots$ By abusing notation, we denote the accuracy of the destination as $S_0^* = 0$. Furthermore, let $\{N^*(t), t \geq 0\}$ be the counting process of the number of messenger nodes with accuracy in the interval $[0,t]$ at time T, defined by $N^*(t) = |\{i : S_i^* \leq t\}|$. In other words, $N^*(t)$ is the number of nodes whose last encounter with the destination was in the time interval $[T-t, T]$. By definition, $\{N^*(t), t \geq 0\}$ is a counting process whose corresponding arrival process is $\{S_i^*, i \geq 1\}$.

It is easy to see that $N^*(T) = N(T)$. The two processes are not necessarily equal for other values of t. However, their distributions are the same, as stated in the following lemma.

Lemma 14: Processes $\{N(t), t \geq 0\}$ and $\{N^*(t), t \geq 0\}$ are stochastically equivalent.

A proof can be found in [19]. The intuition is that, if one reverses the "arrow of time", the last times nodes exit the transmission region around destination become the first times they enter it. This lemma implies that corresponding versions of Lemmas 1, 2 and 3 hold for $N^*(t)$ as well.

Let $I(T)$ be the index of the node that is chosen at a flooding initiated at time T at the origin defined as $I(T) = \arg\min_{i:S_i^*<T} X_i(T)$. Then $p_0(t,T)$ is the density of $S_{I(T)}^*$. Consider $\mathbf{P}\left\{S_{I(T)}^* \geq t\right\}$ where $t > 0$. For $t > 0$, the event $\{S_{I(T)}^* \geq t\}$ implies the event $\{S_{I(T)}^* > 0\}$, which in turn implies that at least one messenger node with accuracy greater than zero exists (namely I). The latter is represented by the event $\{N^*(T) - N^*(0) \geq 1\}$. On the other hand, $\{S_{I(T)}^* \geq t\}$ also implies $\{S_{N^*(T)}^* \geq t\}$, since, by definition, $S_{N^*(T)}^* \geq S_{I(T)}^*$. Hence, for $t > 0$, $\mathbf{P}\left\{S_{I(T)}^* \geq t\right\} \leq \mathbf{P}\left\{S_{N^*(T)}^* \geq t \cap N^*(T) - N^*(0) \geq 1\right\}$ Let $\tilde{N}^*(t) = N^*(t) - N^*(0)$. Then, by Lemmas 14 and 3 $\mathrm{E}\left[N\right]^*(T) = \mu_0(2t) - \mu_0(0)$, where $\mu_0(t)$ is described by eq. (6). The r.h.s. of the above inequality can then be written as $\mathbf{P}\left\{\tilde{S}_{\tilde{N}^*(T)}^* \geq t \cap \tilde{N}^*(T) \geq 1\right\}$ where $\{\tilde{S}_i^*, i \geq 1\}$ is the arrival process corresponding to the counting process $\{\tilde{N}^*(t), t \geq 0\}$. This probability can be written as the expectation of $\mathbf{P}\left\{\tilde{S}_{\tilde{N}_f^*(T)}^* \geq t \cap \tilde{N}_f^*(T) \geq 1\right\}$ over all possible paths f the destination may follow in the interval $[0,T]$. By Lemmas 14 and 1, $\tilde{N}_f^*(t)$ is Poisson distributed with expectation $\mu_f(t) - \mu_f(0)$. The above probability is equal to

$$\sum_{n=1}^{\infty} \mathbf{P}\left\{\tilde{S}_n^*) \geq t \mid \tilde{N}_f^*(T) = n\right\} \mathbf{P}\left\{\tilde{N}_f^*(T) = 1\right\}.$$

Since $\tilde{N}_f^*$ is a Poisson process however, we have that

$$\mathbf{P}\left\{\tilde{S}_n^* > t \mid \tilde{N}_f^*(T) = n\right\} = \int_t^T \frac{n(\tilde{\mu}_f(s))^{n-1}}{(\tilde{\mu}_f(T))^n} d\tilde{\mu}_f(s) \quad \text{and}$$

$$\mathbf{P}\left\{\tilde{N}_f^*(T) = n\right\} = e^{\tilde{\mu}_f(T)} \frac{\tilde{\mu}_f(T)^n}{n}.$$

Therefore $\mathbf{P}\left\{\tilde{S}_{\tilde{N}_f^*(T)}^* \geq t \cap \tilde{N}_f^*(T) \geq 1\right\} = 1 - e^{\tilde{\mu}_f(t) - \tilde{\mu}_f(T)}$. Taking the expectation over all paths gives the lemma, noting that $\tilde{\mu}_f(t) - \tilde{\mu}_f(T) = \mu_f(t) - \mu_f(T)$, $\mathrm{E}\left[e^{\mu_f(t)-\mu_f(T)}\right] \geq e^{\mathrm{E}[\mu_f(t)-\mu_f(T)]}$ and that $\mathrm{E}\left[\mu_f(t)\right] = \mu_0(2t)$ by Lemma 3. ■

APPENDIX IV
PROOF OF THEOREM 2

We first note that $p_0(t,T)$ contains a δ_0 term, a "jump" at 0: there is a non zero probability q that the node that we locate will have accuracy zero -in fact, one can lower-bound this by $e^{-\mu_0(2T)}$. The Volterra equation (1) can thus be written as

$$\begin{aligned} Q_0(T) &= G_0(T) + qQ_0(0) + \int_0^T \hat{p}(t,T)Q(t)dt \\ &= G_0(T) + \int_0^T \hat{p}(t,T)Q(t)dt \end{aligned}$$

where $\hat{p}(t,T)$ is $p(t,T)$ without the δ_0 term (i.e. $\hat{p}(t,T) = p(t,T)$ for $t > 0$ and $\int_0^T \hat{p}(t,T)dt = 1 - q$). Working on basic principles of Volterra equations, one can show that the following two lemmas hold. We omit their proofs for reasons of brevity.

Lemma 15: Let $Q_a(T), Q_b(T), T \geq \alpha > 0$ be the solutions of the equations

$$Q_a(T) = G_a(T) + \int_0^T p(t,T)Q_a(t)dt \qquad \text{and}$$

$$Q_b(T) = G_b(T) + \int_0^T p(t,T)Q_b(t)dt$$

respectively, where $G_a(T) \geq 0$, $G_b(T) \geq \epsilon > 0$, $p(t,T) > 0$, and $G_a(T) = O(G_b(T))$. Then $Q_a(T) = O(Q_b(T))$.

This lemma indicates that, to obtain an asymptotic upper bound for $Q(T)$, one may do so by computing such a bound for $G(T)$.

Lemma 16: Let $Q_a(T)$, $Q_b(T)$ be the solutions of the equations

$$Q_a(T) = G(T) + \int_0^T p_a(t,T)Q_a(t)dt \qquad \text{and}$$

$$Q_b(T) = G(T) + \int_0^T p_b(t,T)Q_b(t)dt$$

respectively, where $G(T) \geq 0$, $G'(T) \geq 0$ in $[0, +\infty)$, for all $0 \leq \tau \leq T$

$$\int_\tau^T p_a(t,T)dt \leq \int_\tau^T p_b(t,T)dt < 1, \quad (19)$$

and or any fixed $\tau \geq 0$, $\frac{\partial}{\partial T}\int_\tau^T p_b(t,T)dt \geq 0$ for all T in $[\tau, +\infty)$. Then $Q_a(T) \leq Q_b(T)$.

The above formalizes the concept that we stated in Section VI-B regarding stochastic domination. Lemma 5 indicates that $G_0(T) = O(\log(T+2))$ and makes Lemma 15 applicable. Furthermore, Lemma 6 allows us to make use of Lemma 16. Hence $Q_0(T)$ is asymptotically upper-bounded by the solution of

$$Q(T) = \log(T+2) + \int_0^T \exp(\mu(t) - \mu(T))Q(t)d\mu(t).$$

The kernel of this Volterra equation is separable (see [18]) and the solution can be computed. It is equal to

$$\begin{aligned} Q(T) &= \log(T+2) + \int_0^T \log(t+2)d\mu(t)) \\ &\leq \log(T+2)(1+\mu(T)) = O(T) \end{aligned}$$

since $\mu(t) = \mu_0(2t) = O(2\pi vt/\log vt)$. ■

Appendix V
Proof of Lemma 9

Lemma 7 implies that, if $A(T) = \Theta(\log^{1+2\alpha} T)$, the expected one-step flooding area $G_1(T)$ is $O(\log^{1+2\alpha} T)$. Using this result, one can show that $Q_1(T) \leq Q^*(T)O(\log^{1+2\alpha} T)$, where $Q^*(T)$ is the expected number of flooding steps the protocol makes, given by:

$$Q^*(T) = 1 + \int_0^T Q^*(t)p_1(t,T)dt. \tag{20}$$

From Lemma 8, we know that, if $A(T) = \Theta(\log^{1+2\alpha} T)$, for every $0 < \epsilon < 1$ there exists a $T_0 > 0$ such that for all $T > T_0$

$$\int_0^{T-\tau(T)} p_1(t,T)dt \geq 1 - \epsilon, \tag{21}$$

where $\tau(T) = \beta\frac{T}{\log^\alpha T}$, and $0 < \beta < 1$. Note that $F(T) = T - \tau(T)$ and $\tau(T)$ are strictly increasing for large enough T. One can thus take T_0 such that (21) holds and $F(T)$, $\tau(T)$ are strictly increasing in $[T_0, \infty)$.

Let $M = \sup_{[0,T_0]} Q^*(T)$. Based on the fact that the probability that the destination is located through flooding in LER_1 is positive, one can show that $M < \infty$. Similarly one can show that $\sup_{[T_1,T_2]} Q^*(T) < \infty$ for any T_1, T_2. Using M, we construct a function $S(T)$ such that $Q^*(T) \leq S(T)$ for all T. Let $S(T)$ be the solution of the set of equations:

$$S(T) = \frac{1}{1-\epsilon} + S(T - \tau(T)), \qquad T > T_0 \tag{22a}$$

$$S(T) = M, \qquad 0 \leq T \leq T_0 \tag{22b}$$

We will show that $Q^*(T) \leq S(T)$ for all T.

Note first that $S(T)$ has the following form: As $F(T) = T - \tau(T)$ is strictly increasing in $[T_0, \infty)$, a strictly increasing inverse F^{-1} exists in this interval. Furthermore, $F(T) < T$, hence $F^{-1}(T) > T$. We define the sequence $T_1 = F^{-1}(T_0)$, $T_2 = F^{-1}(T_1)$,... which is strictly increasing. By definition, for $i \geq 1$,

$$T_i - T_{i-1} = \tau(T_i) \tag{23}$$

and, as $\tau(T)$ is strictly increasing in $[T_0, +\infty)$, the sequence $\{T_i\}$ is divergent and $T_i \to +\infty$. The solution of (22) can then be written as $S(T) = M$ for $T \in [0, T_0]$, $S(T) = M + 1/(1-\epsilon)$, for $T \in (T_0, T_1]$,..., $S(T) = M + i/(1-\epsilon)$ for $T \in (T_{i-1}, T_i]$. By definition of M, $Q^*(T) \leq S(T)$ for all $T \leq T_0$. Consider some $T \in (T_0, T_1]$. Then $S(T) = 1/(1-\epsilon) + M$ and, by (20),

$$\begin{aligned} Q^*(T) &\leq 1 + \sup_{[0,T-\tau(T)]} Q^*(t) \int_0^{T-\tau(T)} p_1(t,T)dt + \dots \\ &\quad \sup_{[T-\tau(T),T]} Q^*(t) \int_{T-\tau(T)}^T p_1(t,T)dt \\ &\leq 1 + M \int_0^{T-\tau(T)} p_1(t,T)dt + \dots \\ &\quad \sup_{[0,T_1]} Q^*(t) \int_{T-\tau(T)}^T p_1(t,T)dt \end{aligned}$$

On the other hand, $\sup_{[0,T_1]} Q^*(t) \geq \sup_{[0,T_0]} Q^*(t) = M$. Furthermore, by (21), $\int_0^{T-\tau(T)} p_1(t,T)dt \geq 1 - \epsilon$ and thus

$$Q^*(T) \leq 1 + (1-\epsilon)M + \epsilon \sup_{[0,T_1]} Q^*(t).$$

The above inequality holds for all $T \in (T_0, T_1]$. It trivially holds for all $T \in [0, T_0]$ as well. Hence

$$\sup_{[0,T_1]} Q^*(t) \leq 1 + (1-\epsilon)M + \epsilon \sup_{[0,T_1]} Q^*(t)$$

which gives us that $Q^*(T) \leq S(T)$ in $(T_0, T_1]$. Finally, $Q^*(T) \leq S(T)$ for all $T > T_1$ can be shown by induction, by setting $M' = \sup_{[0,T_i]} Q^*(t) \leq M + i/(1-\epsilon)$ and constructing as above an $S' \leq S$ which upper-bounds $Q^*(T)$ in $(T_i, T_{i+1}]$.

As $Q_1(T) \leq Q^*(T)O(\log^{1+2\alpha}(T))$ and $Q^*(T) \leq S(T)$ for all T, it suffices to show that $\limsup \frac{S(T)}{\log^{1+\alpha} T}$ is upper-bounded by a constant. To see this, note that

$$\begin{aligned} S(T_i) - M &= \frac{i}{1-\epsilon} = \frac{1}{1-\epsilon}\sum_{k=1}^{i} 1 = \frac{1}{1-\epsilon}\sum_{k=1}^{i}\frac{T_k - T_{k-1}}{T_k - T_{k-1}} \\ &= \frac{1}{1-\epsilon}\sum_{k=1}^{i}(T_k - T_{k-1})\frac{1}{\tau(T_k)}, \qquad \text{by (23)} \\ &\leq \frac{1}{1-\epsilon}\int_{T_0}^{T_i}\frac{1}{\tau(t)}dt = \frac{1}{1-\epsilon}\int_{T_0}^{T_i}\frac{\log^\alpha t}{\beta t}dt \end{aligned}$$

as $\tau(T) = \beta\frac{T}{\log^\alpha T}$ is increasing in $[T_0, +\infty)$. For any $T > T_0$, let $\hat{T} = T_i$ such that $T \in (T_i, T_{i+1}]$. Then

$$\begin{aligned} \limsup_{T\to\infty}\frac{S(T)}{\log^{1+\alpha} T} &\leq \limsup_{T\to\infty}\frac{S(\hat{T}) + \frac{1}{1-\epsilon}}{\log^{1+\alpha}\hat{T}} = \lim_{i\to\infty}\frac{S(T_i) + \frac{1}{1-\epsilon}}{\log^{1+\alpha} T_i} \\ &\leq \lim_{i\to\infty}\frac{\frac{1}{(1-\epsilon)\beta}\int_{T_0}^{T_i}\frac{\log^\alpha t}{t}dt + M + \frac{1}{1-\epsilon}}{\log^{1+\alpha} T_i} \\ &\leq \frac{1}{(1-\epsilon)\beta} \end{aligned}$$

as $\int_{T_0}^{T_i}\frac{\log^\alpha t}{t}dt \leq \log^\alpha T_i \int_{T_0}^{T_i}\frac{dt}{t}$. ■