A Backlog-Based CSMA Mechanism to Achieve Fairness and Throughput-Optimality in Multihop Wireless Networks

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***Abstract*— We propose and analyze a distributed backlog-based CSMA policy to achieve fairness and throughput-optimality in wireless multihop networks. The analysis is based on a CSMA fixed point approximation that is accurate for large networks with many small flows and a small sensing period.**

I. Introduction

We propose and analyze a backlog-based CSMA policy to achieve fairness and throughput-optimality in wireless multihop networks with primary interference constraints. Our analysis uses a novel CSMA fixed point approximation that was presented in [1]. The proposed CSMA policy is simple and can easily been implemented in a distributed manner as it uses only local information, and does not require any exchange of information (such as queue-lengths or congestion signals) between nodes.

This paper adds to a new understanding of CSMA policies that shows CSMA policies can achieve fairness and throughput optimality. Related to our work, Jiang and Walrand considered in [2] CSMA policies for the idealized situation of instantaneous channel feedback. The assumption of instantaneous channel feedback eliminates packet collisions which simplifies the analysis. Under this assumption Jiang and Walrand derived a dynamic CSMA policy that, combined with rate control, achieves throughput-optimality while satisfying a given fairness criterion. Similar results have been independently derived by Shah and Sreevastsa in [3] in the context of optical networks. While these results are obtained for an idealized model of CSMA policies, they suggest that CSMA policies might be throughput-optimal. In this paper, we confirm this conjecture for wireless networks with primary interference constraints. For a related work on scheduling in wireless multihop networks we refer to [1] and [2]

II. System Model

We consider a fixed wireless network composed of a set $\mathcal{N}$ of nodes with cardinality N, and a set $\mathcal{L}$ of directed links with cardinality L. A directed link $(i, j) \in \mathcal{L}$ indicates that node i is able to send data packets to node j. We assume that the rate of transmission is the same for all links and all packets are of a fixed length. Throughout the paper we rescale time such that the time it takes to transmit one packet is equal to one time unit.

For a given node $i \in \mathcal{N}$, let $\mathcal{U}_i := \{j \in \mathcal{N} : (i, j) \in \mathcal{L}\}$ be the set of upstream nodes, i.e. the set containing all nodes which can receive packets from i. Similarly, let $\mathcal{D}_i := \{j \in \mathcal{N} : (j, i) \in \mathcal{L}\}$ be set of downstream nodes, i.e. the set containing all nodes j from which i can receive packets. Collectively, we denote the set of all the neighbors of node i as $\mathcal{N}_i := \mathcal{U}_i \cup \mathcal{D}_i$. Also, we let $\mathcal{L}_i := \{(i, j) : j \in \mathcal{D}_i\}$ be the set of outgoing links from node i, i.e. the set of all links from node i to its upstream nodes $\mathcal{U}_i$.

Throughout the paper, we assume that $\mathcal{U}_i = \mathcal{D}_i$, for all $i \in \mathcal{N}$ so that we have $\mathcal{U}_i = \mathcal{D}_i = \mathcal{N}_i$, for each $i \in \mathcal{N}$. This assumption simplifies the notation as we can use a single set $\mathcal{N}_i$ to represent both $\mathcal{D}_i$ and $\mathcal{U}_i$. Our analysis can be extended to the more general case requiring only notational changes.

We assume that there is a fixed set of routes $\mathcal{R}$ which defines the possible traffic paths in the network. For a given route $r \in \mathcal{R}$, let s_r be its source node and d_r be its destination node. Furthermore, we use the convention that r is given by the set of links that the route traverses, i.e. we let

$$r = \{(s_r, i), (i, j), \cdots, (v, w), (w, d_r)\}$$

be the set of links traversed by the route. We allow several routes to be defined for a given source and destination pair (s, d), $s, d \in \mathcal{N}$.

In the following, we describe a network by the tuple $(\mathcal{L}, \mathcal{R})$ since the set of nodes $\mathcal{N}$ can be derived from the set $\mathcal{L}$.

We focus on networks under the well-known *primary interference*, or *node exclusive interference*, model [4].

Definition 1 (Primary Interference Model): A packet transmission over link $(i, j) \in \mathcal{L}_i$ is successful if only if within the transmission duration there exist no other activity over any other link $(m, n) \in \mathcal{L}$ which shares a node with (i, j). $\diamond$

The primary interference model applies, for example, to wireless systems where multiple frequencies/codes are available (using FDMA or CDMA) to avoid interference, but each node has only a single transceiver and hence can only send to or receive from one other node at any time.

We characterize the network traffic with a rate vector $\lambda := \{\lambda_r\}_{r \in \mathcal{R}}$ where λ_r, $\lambda_r \geq 0$, is the mean packet arrival rate in packets per unit time on route $r \in \mathcal{R}$.

Given the rate vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$, we let

$$\lambda_{(i,j)} := \sum_{r:(i,j)\in\mathcal{R}_r} \lambda_r, \qquad (i,j) \in \mathcal{L}, \tag{1}$$

be the mean packet arrival rate to link (i,j). Similarly, we let

$$\Lambda_i := \sum_{j\in\mathcal{N}_i} \big[\lambda_{(i,j)} + \lambda_{(j,i)}\big], \qquad i \in \mathcal{N}. \tag{2}$$

be the mean packet arrival rate to node $i \in \mathcal{N}$.

With each route r we associate a utility function U_r that depends on the rate λ_r allocated to route r. We make the following assumption.

Assumption 1: The utility function $U_r(\lambda_r)$, $r \in \mathcal{R}$, is increasing and strictly concave with $U_r(0) = 0$ and $\lim_{\lambda_r\to\infty} U_r'(\lambda_r) = 0$.

III. Throughput-Optimality and Fairness

Consider a fixed network $(\mathcal{L}, \mathcal{R})$ with traffic vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$. A scheduling *policies* π then defines the rules that are used to schedule packet transmissions on each link $(i,j) \in \mathcal{L}$. In the following we focus on policies π that have a well-defined link service rates as a function of the rate vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$.

Definition 2: (Service Rate): Consider a fixed network $(\mathcal{L}, \mathcal{R})$. The link service rate $\mu^{\pi}_{(i,j)}(\lambda)$, $(i,j) \in \mathcal{L}$, of policy π for the traffic vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$ is the fraction of time node i spends successfully transmitting packets on link (i,j) under π and λ, i.e. the fraction of time node i sends packets over link (i,j) that do not experience a collision.

Let $\mathcal{P}$ be the class of all policies π that have well-defined link service rates. Note that this class contains a broad range of scheduling policies, including dynamic policies such as queue-length-based policies [5], as well as noncausal policies that know the future arrival of the flows. We then define network stability as follows.

Definition 3 (Stability): For a given network $(\mathcal{L}, \mathcal{R})$, let $\mu^{\pi}(\lambda) = \{\mu^{\pi}_{(i,j)}(\lambda)\}_{(i,j)\in\mathcal{L}}$ the link service rates of policy π, $\pi \in \mathcal{P}$, for the rate vector $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$. We say that policy π stabilizes the network for λ if

$$\lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda), \qquad \text{for all } (i,j) \in \mathcal{L}. \qquad \diamond$$

This commonly used stability criteria [5] requires that for each link (i,j) the link service rate $\mu^{\pi}_{(i,j)}(\lambda)$ is larger than the arrival rate $\lambda_{(i,j)}$. The capacity region of a network $(\mathcal{L}, \mathcal{R})$ is then defined as follows.

Definition 4: (Capacity Region) For a given a network $(\mathcal{L}, \mathcal{R})$, the capacity region $\mathcal{C}$ is equal to the set of all traffic vectors $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$ such that there exists a policy $\pi \in \mathcal{P}$ that stabilizes the network for λ, i.e. we have

$$\mathcal{C} = \{\lambda \geq 0 : \exists \pi \in \mathcal{P} \text{ with } \lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda),\ \forall (i,j) \in \mathcal{L}\}.$$

We say that an optimal solution $\lambda^* = \{\lambda^*_r\}_{r\in\mathcal{R}}$ to the maximization problem

$$\max{}_{\lambda\in\mathcal{C}} \sum_{r\in\mathcal{R}} U_r(\lambda_r)$$

achieves *fairness and throughput-optimality*.

IV. CSMA Scheduling Policies

In the following, we study whether CSMA policies can achieve fairness and throughput-optimality as defined above.

A. CSMA Policies

We consider CSMA policies that are given by a transmission attempt probability vector $\mathbf{p} = (p_{(i,j)})_{(i,j)\in\mathcal{L}} \in [0,1]^L$ and a sensing period (or idle period) $\beta > 0$. The policy works as follows: each node, say i, senses the activity on its outgoing links $l \in \mathcal{L}_i$. If link $(i,j) \in \mathcal{L}_i$ has been idle for a duration of an idle period β, i.e. node i and j have not sent, or received, a packet for β time units, then i starts a transmission of a single packet on link (i,j) with probability $p_{(i,j)}$, independent of all other events in the network. If node i does not start a packet transmission, then link (i,j) has to remain idle for another period of β time units before i again has the chance to start a packet transmission. Thus, the epochs at which node i has the chance to transmit a packet on link (i,j) are separated by periods of length β during which link (i,j) is idle, and the probability that i starts a transmission on link (i,j) after the link has been idle for β time units is equal to $p_{(i,j)}$,

We assume that packet transmission attempts are made according to above description regardless of the availability of packets at the transmitter. In the event of a transmission decision in the absence of packets, the transmitting node transmits a *dummy* packet, which is discarded at the receiving end of the transmission.

The duration of an idle period β is again given relative to the length of a packet transmission which is assumed to take one unit time, i.e. if the length of an sensing period is L_i seconds and the length of a packet transmission is L_p seconds, then we have $\beta = L_i/L_p$. For a fixed L_i, the duration of an idle period β will become small if we increase the packet lengths, and hence the packet transmission delay L_p.

Given the length of an sensing period β, in the following we will sometimes refer to $\mathbf{p}$ as the CSMA policy.

B. Achievable Rate Region of CSMA Policies

One can show [1] that a CSMA policy $\mathbf{p}$ has a well-defined link service rate vector, i.e. CSMA policies are contained in the set $\mathcal{P}$. Given a sensing period β, we denote with $\mu(\mathbf{p}) = \{\mu_{(i,j)}\}_{(i,j)\in\mathcal{L}}$ the link service rate vector under the CSMA policy $\mathbf{p}$. Note that for a given β, the link service rate under a CSMA policy depend only on the transmission attempt probability vector $\mathbf{p}$, but not on the arrival rates λ. The achievable rate region of CSMA policies is then given as follows.

Definition 5 (Achievable Rate Region of CSMA Policies): For a given network $(\mathcal{L}, \mathcal{R})$ and a given sensing period β, the *achievable rate region of CSMA policies* is given by the set of rate vectors $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$ for which there exists a CSMA policy $\mathbf{p}$ that stabilizes the network for λ, i.e. we have that $\lambda_{(i,j)} < \mu_{(i,j)}(\mathbf{p})$, $(i,j) \in \mathcal{L}$.

Let $\mathcal{C}_{CSMA}$ be the achievable rate region of CSMA policies. We are interested in the following questions: (a) can we characterize $\mathcal{C}_{CSMA}$, (b) how close is C_{CSMA} to the capacity region $\mathcal{C}$, (c) can we characterize and compute solutions to $\max_{\lambda \in \mathcal{C}_{CSMA}} \sum_{r \in \mathcal{R}} U_f(\lambda_r)$, and (d) can we obtain solutions to $\max_{\lambda \in \mathcal{C}_{CSMA}} \sum_{r \in \mathcal{R}} U_f(\lambda_r)$ using a distributed mechanism.

In [1], it was shown that the achievable rate region of CSMA policies is equal to the capacity region $\mathcal{C}$ under the limiting regime of large networks with many small flows and a small sensing period β. This result was obtained using a novel CSMA fixed point approximation that extends the well-known infinite node approximation for single-hop networks to the case of multihop networks. In this paper we using the CSMA fixed point approximation to address above question (c) and (d). In the next section, we provide a summary of the results given [1] regarding the CSMA fixed point approximation.

V. CSMA Fixed Point Approximation

In this section, we present the CSMA fixed point approximation, where we first briefly review the infinite node approximation for single-hop networks. In the following we will use τ to denote the services rates obtained under our analytical formulations that we use to approximate the actual service rates μ under a CSMA policy as defined in Section IV.

A. Infinite Node Approximation for Single-Hop Networks

Consider a single-hop network where N nodes share a single communication channel, i.e. where nodes are all within transmission range of each other. In this case, a CSMA policy is given by the vector $\mathbf{p} = (p_1, \cdots, p_N) \in [0,1]^N$ where p_n is the probability that node n starts a packet transmission after an idle period of length β [6].

The network throughput, i.e. the fraction of time the channel is used to transmit packets that do not experience a collision, can then be approximated by (see for example [6])

$$\tau(G(\mathbf{p})) = \frac{G(\mathbf{p}) e^{-G(\mathbf{p})}}{\beta + 1 - e^{-G(\mathbf{p})}} \tag{3}$$

where

$$G(\mathbf{p}) = \sum_{n=1}^{N} p_n.$$

Note that $G(\mathbf{p})$ captures the expected number of transmissions attempt after an sensing period β for a given CSMA policy $\mathbf{p}$.

This well-known approximation is based on the assumption that a large (infinite) number of nodes share the communication channel. It is asymptotically accurate as the number of nodes N becomes large and each node makes a transmission attempt with a probability p_n, $n \in \mathcal{N}$ that approaches zero while the offered load $G = \sum_{n=1}^{N} p_n$ stays constant (see for example [6]).

The following results are well-known. For $\beta > 0$, we have that $\tau(G) < 1$, $G \geq 0$, and for $G^+(\beta) = \sqrt{2\beta}$, $\beta > 0$, we have that $\lim_{\beta \downarrow 0} \tau(G^+(\beta)) = 1$.

Using (3), the service rate $\mu_n(\mathbf{p})$ of node n under a given static CSMA policy $\mathbf{p}$ can be approximated by

$$\tau_n(\mathbf{p}) \quad = \quad \frac{p_n e^{-G(\mathbf{p})}}{1 + \beta - e^{-G(\mathbf{p})}}, \qquad n = 1, ..., N. \tag{4}$$

In the above expression, p_n is the probability that node n starts a packet transmission if the channel has been idle for a sensing period of duration β, and $e^{-G(\mathbf{p})}$ characterizes the probability that this attempt is successful, i.e. the attempt does not collide with an attempt by any other node.

Similarly, the fraction of time that the channel is idle can be approximated by

$$\rho(\mathbf{p}) = \rho(G(\mathbf{p})) = \frac{\beta}{\beta + 1 - e^{-G(\mathbf{p})}}, \tag{5}$$

where we have that $\lim_{\beta \downarrow 0} \rho(G^+(\beta)) = 0$.

B. CSMA Fixed Point Approximation for Multihop Networks

We extend the above approximation for single-hop networks to multihop networks as follows.

For a given a sensing period β, we approximate the fraction of time $\rho_i(\mathbf{p})$ that node i is idle under the CSMA policy $\mathbf{p}$ by the following fixed point equation,

$$\rho_i(\mathbf{p}) = \frac{\beta}{\beta + 1 - e^{-G_i(\mathbf{p})}}, \qquad i = 1, \cdots, N, \tag{6}$$

where

$$G_i(\mathbf{p}) = \sum_{j \in \mathcal{N}_i} \left[p_{(i,j)} + p_{(j,i)}\right] \rho_j(\mathbf{p}), \qquad i = 1, \cdots, N. \tag{7}$$

which approximately measures the transmission attempt rate of node i given that it is idle. We refer to the above fixed point equation as the *CSMA fixed point* equation, and to a solution $\rho(\mathbf{p}) = (\rho_1(\mathbf{p}), \cdots, \rho_N(\mathbf{p}))$ and $G(\mathbf{p}) = (G_1(\mathbf{p}), \cdots, G_N(\mathbf{p}))$ to the fixed point equation as a *CSMA fixed point*.

The intuition behind the CSMA fixed point equation Eq. (6) and Eq. (7) is as follows: suppose that the fraction of time that node i is idle under the static CSMA policy $\mathbf{p}$ is equal to $\rho_i(\mathbf{p})$, and suppose that the times when node i is idle are independent of the processes at all other nodes. If node i has been idle for β time units, i.e. node i has not received or transmitted a packet for β time units, then node i can start a transmission attempt on link (i,j), $j \in \mathcal{N}_i$, only if node j also has been idle for an idle period of β time units. Under the above independence assumption, this will be (roughly) the case with probability $\rho_j(\mathbf{p})$, and the probability that node i start a packet transmission on the link (i,j), $j \in \mathcal{N}_i$, given that it has been idle for β time units is (roughly) equal to $p_{(i,j)}\rho_j(\mathbf{p})$. Similarly, the probability that node $j \in \mathcal{N}_i$ starts a packet transmission on the link (j,i) after node i has been idle for β time units is (roughly) equal to $p_{(j,i)}\rho_j(\mathbf{p})$. Hence, the expected number of transmission attempts that node i makes or receives, after it has been idle for β time units is (roughly) given by Eq. (7). Using Eq. (5) of Section V-A, the fraction of time that node i is idle under $\mathbf{p}$ can then be approximated by Eq. (6).

For a given an sensing period β, we can then use the CSMA fixed point $G(\mathbf{p})$ for a policy $\mathbf{p}$ to approximate the link service rate $\mu_{(i,j)}(\mathbf{p})$ under a CSMA policy $\mathbf{p}$ by

$$\tau_{(i,j)}(\mathbf{p}) = \frac{p_{(i,j)}\rho_j(\mathbf{p})e^{-G_i(\mathbf{p})}}{1+\beta-e^{-G_i(\mathbf{p})}}e^{-G_j(\mathbf{p})}, \quad (i,j) \in \mathcal{L}. \quad (8)$$

Note that the above equation is similar to (4) where $p_{(i,j)}\rho_j(\mathbf{p})$ captures the probability that node i starts a packet transmission on link (i,j) if node i has been idle for β time units, and $\exp\left[-(G_i(\mathbf{p})+G_j(\mathbf{p}))\right]$ is the probability that this attempt is successful, i.e. the attempt does not overlap with an attempt by any other node to capture a link that has an endpoint in common with link (i,j).

There are two important questions regarding the CSMA fixed point approximation. First, one needs to show that the CSMA fixed point is well-defined, i.e. that there always exists a unique CSMA fixed point. In [1] it was shown that this is indeed the case. Second, one would like to know how accurate the CSMA fixed point approximation is. In [1] it was shown that the CSMA fixed point approximation is asymptotically accurate for large networks with many small flows and a small sensing period β.

C. Approximate Achievable Rate Region

In this section we use the CSMA fixed point approximation to characterize the achievable rate region of CSMA policies.

Consider a network $(\mathcal{L},\mathcal{R})$ with sensing time $\beta > 0$ as described in Section II, and let $\Gamma(\beta)$ be given by

$$\Gamma(\beta) = \left\{\lambda = \{\lambda_r\}_{r\in\mathcal{R}} | \Lambda_i < \tau(G^+(\beta))e^{-(G^+(\beta))},\ \ i \in \mathcal{N}\right\},$$

where $G^+(\beta) = \sqrt{2\beta}$ and $\tau(G^+(\beta)$ are as defined in Section V-A, and

$$\Lambda_i = \sum_{j\in\mathcal{N}_i}\left[\lambda_{(i,j)} + \lambda_{(j,i)}\right], \qquad i \in \mathcal{N},$$

is as defined in Section II.

The next result states that for a network $(\mathcal{L},\mathcal{R})$ with sensing time $\beta > 0$ the achievable rate region of CSMA policies under the CSMA fixed point approximation contains the set $\Gamma(\beta)$.

Theorem 1: Given a network $(\mathcal{L},\mathcal{R})$ with sensing time $\beta > 0$, for every $\lambda \in \Gamma(\beta)$ there exists a CSMA policy $\mathbf{p}$ such that

$$\lambda_{(i,j)} < \tau_{(i,j)}(\mathbf{p}), \qquad (i,j) \in \mathcal{L},$$

where $\tau_{(i,j)}(\mathbf{p})$ is the service rate for link (i,j) under the CSMA fixed point approximation as given by Eq. (8).

We refer to [1] for a proof of Theorem 1. The proof of Theorem 1 given in [1] is constructive in the sense that given a rate vector $\lambda \in \Gamma(\beta)$, it constructs a CSMA policy $\mathbf{p}$ such that $\lambda_{(i,j)} < \tau_{(i,j)}(\mathbf{p})$, $(i,j) \in \mathcal{L}$. We will use this result in the next section.

Note that from Section V-A, we have that

$$\lim_{\beta\to 0} G^+(\beta) = 0$$

and

$$\lim_{\beta\to 0} \tau(G^+(\beta)) = 1$$

Using these results, we obtain that

$$\lim_{\beta\downarrow 0} \Gamma(\beta) = \{\lambda = \{\lambda_r\}_{r\in\mathcal{R}} | \Lambda_i < 1 \ \ i = 1,\cdots,N\}.$$

Furthermore, note that any rate vector λ for which there exists a node i with $\Lambda_i \geq 1$ cannot be stabilized, as the service rate at each node is upper-bounded by 1. Hence, the above result suggests that for network with a small sensing time the achievable rate region of static CSMA policies is equal to the capacity region. In [1] it was shown that this is indeed true (i.e. the characterization of the achievable rate region using the CSMA fixed point approximation is accurate) for the limiting regime of large networks with many small flows and a small sensing period β. We will use this limiting regime for our analysis in Section VI and VIII-E.

VI. A Centralized Mechanism for Multihop Networks

Consider a network $(\mathcal{L},\mathcal{R})$ with sensing time $\beta > 0$ as described in Section II. For

$$C(\beta) = \tau(G^+(\beta))e^{-(G^+(\beta))},$$

let λ^* be the optimal rate vector for the utility maximization problem

$$\begin{aligned} &\max . \sum_{r\in\mathcal{R}} U_r(\lambda_r) \\ &s.t. \left[\sum_{r:(i,j)\in r}\lambda_r + \sum_{r:(j,i)\in r}\lambda_r\right] \leq C(\beta)(1-\delta),\ i \in \mathcal{N} \\ &\quad \lambda_r \geq 0, r \in \mathcal{R}, \end{aligned}$$

where $\delta > 0$ is a small positive constant. Note that

$$\lambda^* \in \Gamma(\beta),$$

and using the proof of [1] we can construct a CSMA policy $\mathbf{p}^*$ that supports λ^* in the sense that

$$\lambda_{(i,j)} < \tau_{(i,j)}(\mathbf{p}^*), \qquad (i,j) \in \mathcal{L}.$$

As we have that

$$\lim_{\beta\to 0} C(\beta) = 1,$$

it follows that under the CSMA fixed point approximation this procedure can be used to construct a CSMA policy that achieves fairness and throughput-optimality in wireless multihop networks in the limiting regime as β becomes small.

While the above result was obtained under the CSMA fixed point approximation, we next consider actual link rates $\mu_{(i,j)}(\mathbf{p}^*)$ under the CSMA policy λ^* that we obtain through the above procedure, and study whether we have

$$\lambda_{(i,j)} < \mu_{(i,j)}(\mathbf{p}^*), \qquad (i,j) \in \mathcal{L},$$

and $\mathbf{p}^*$ indeed stabilizes the network for the rate vector λ^*. The next result states that this is indeed the case under the limiting regime of large network with many small flows and a small sensing period β.

Consider a sequence of networks with N nodes and sensing period $\beta^{(N)}$ such that

$$\lim_{N\to\infty} N\beta^{(N)} = 0.$$

Furthermore, let $\mathcal{R}^{(N)}$ be the set of routes and let $\mathcal{N}_i^{(N)}$, $i \in \mathcal{N}^{(N)}$, the neighbor nodes of node i in the network of size N. Let then $\lambda^{(N)}$ be the optimal solution to

$$\begin{aligned} \max. & \sum_{r\in\mathcal{R}^{(N)}} U_r(\lambda_r) \\ s.t. & \sum_{r:(i,j)\in r} \lambda_r + \sum_{r:(j,i)\in r} \lambda_r \leq C(\beta^{(N)})(1-\delta),\ i \in \mathcal{N}^{(N)} \\ & \lambda_r \geq 0, r \in \mathcal{R}^{(N)}. \end{aligned} \tag{9}$$

Then we have the following result.

Proposition 1: If for the optimal solutions $\lambda^{(N)}$, $N \geq 1$,to Eq. (9 we have that

$$\lim_{N\to\infty} \left(\max_{(i,j)\in\mathcal{R}^{(N)}} \lambda_{(i,j)}^{(N)} \right) = 0,$$

then we obtain that

$$\lim_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\mu_{(i,j)}(\mathbf{p}^{(N)})}{\lambda_{(i,j)}^{(N)}} \right) > 1.$$

The above result states that if the network serves many small flows, then for large N the CSMA policy $\mathbf{p}^{(N)}$ will indeed stabilize the network for the rate vector $\lambda^{(N)}$, and hence CSMA policies achieve fairness and throughput optimality for large networks with many small flows as we have that $\lim_{\beta\to 0} C(\beta) = 1$.

VII. A Distributed Mechanism for Single-Hop Networks

The previous section provides a centralized algorithm for finding a CSMA policy that (asymptotically) achieves fairness and throughput-optimality. Next, we derive a distributed mechanism to achieve this. In this section, we first focus on the special case of a single-hop networks in order to obtain insight into the structure of a distributed mechanism. In the next section, we extend the analysis to multihop networks.

A. Schedulers that Implement a Distributed Queue

Consider a single-cell ad hoc network where a set of N nodes are within transmission range of each other, and suppose that each node has a single buffer. Let λ_n, $n = 1, ..., N$, be the packet arrival rate to the buffer at node n. Consider a scheduler $\pi \in \mathcal{P}$, and let D_n be the expected delay of a packet at node n, and let P_n be the probability that a packet is dropped at node n due to a buffer-overflow under π. For

$$\lambda = \sum_{n=1}^{N} \lambda_n,$$

let $X(\lambda)$ be the network throughput under the network arrival rate λ under π. We are interested in schedulers with the following property.

Property 1: For a single-cell wireless network consisting of nodes $n = 1, ..., N$, we say that a scheduler π implements a distributed buffer with service rate μ if the following is true.

(a) The expected delay D_n under π is identical at all nodes, i.e. we have $D_n = D$, $n = 1, ..., N$.
(b) The packet-drop probability P_n under π is identical at all nodes, i.e. we have $P_n = P$, $n = 1, ..., N$.
(c) The throughput $X(\lambda)$ under π is an non-decreasing function in λ with $\lim_{\lambda\to\infty} X(\lambda) = \mu$.

The above property imply that a scheduler serves packets as if the network traffic shares a common buffer with service rate μ, i.e. all packets entering the network should experience the same expected delay and the same probability of being dropped.

B. Centralized Scheduler

We first consider a centralized scheduler that satisfies Property 1. We assume that the scheduler has perfect information about the backlog at each node, but does not have any knowledge about the packet arrival rates.

Algorithm 1: Consider a single-cell wireless network with N nodes. If at least one buffer has a packet ready to be transmitted and there is currently no packet being transmitted, then initiate a new transmission by scheduling node n with probability

$$q_n = \frac{b_n}{B}, \qquad n = 1, ..., N,$$

where b_n, $n = 1, ..., N$, is the current backlog at node n and

$$B = \sum_{n=1}^{N} b_n$$

is the total backlog over all nodes. If node n is scheduled, then it will transmit the packet at the head of its local queue.

The above algorithm schedules nodes proportionally to their current backlog, hence nodes with a high arrival rate (and a higher backlog) tend to be scheduled more often resulting. We have the following result.

Lemma 1: Consider a single-cell wireless network where each node has an infinite buffer, and suppose that packets arrive to node n according to an independent Poisson process with rate λ_n, and that packet service times are independently and exponentially distributed with mean $\frac{1}{\mu}$. Then Algorithm 1 implements a distributed buffer with rate μ, i.e. the expected delay D is equal to the expected delay at a $M/M/1$ queue with arrival rate $\lambda = \sum_{n=1}^{N} \lambda_n$ and service rate μ.

Lemma 1 states that when the packet arrival process is Poisson and the service rates are exponentially distributed, then the above scheduler satisfies Property 1. Lemma 1 can be proved using Little's theorem [6].

C. Distributed Scheduler

The above centralized algorithm suggests that the probability that a node is scheduled should depend on the current backlog at this node. Using this insight, we consider a distributed algorithm which implements a scheduler with backlog-dependent transmission probabilities.

Let q, $0 < q < 1$, be a constant which is assumed to be small. Each node n, $n = 1, ..., N$ uses the following algorithm to schedule its packet transmissions.

Before each transmission attempt, node n senses whether the channel is idle (no other node is currently transmitting). If the channel has been sensed to be idle for a duration β time units, then the node makes a transmission attempt with probability $q_n = \min\{1-\epsilon, qb_n\}$, where b_n is the current backlog at node n and $\epsilon > 0$ is a small constant to ensure that the attempt probability is strictly smaller than 1.

The above algorithm implements a CSMA policy with backlog-dependent transmission probabilities: the larger the backlog at a node, the more aggressive a node will be in making a transmission attempt. Below we characterize the throughput under this algorithm.

Suppose that the current backlog at node n is equal to b_n such that node n makes transmission attempts with probability qb_n. The expected number of transmission attempts after an idle slot is then given by

$$G = qB \tag{10}$$

where $B = \sum_{n=1}^{N} b_n$ is the total backlog over all nodes. Using the infinite-node approximation of Section V-A, the instantaneous throughput $\tau(G)$ is given by

$$\tau(G) = \frac{Ge^{-G}}{\beta + (1 - e^{-G})}, \qquad G \geq 0. \tag{11}$$

D. Active Queue Management

Eq. (11) states that the throughput under the above backlog-based CSMA policy depends on $G = qB$ where B backlog over all nodes. We use this observation as follows to achieve a high throughput. Consider an active queue management mechanism that randomly drops incoming packets in order to keep the expected number of transmission attempts G at the desired level G^*. We let the probability that a new packet is dropped (called the packet-drop probability $p(u)$) depend on a congestion signal u.

Consider the packet-drop probability $p(u)$ given by

$$p(u) = \begin{cases} \kappa u, & 0 \leq u \leq 1/\kappa, \\ 1, & u > 1/\kappa, \end{cases}$$

where $\kappa > 0$ characterizes the slope of the of the function $p(u)$. The congestion signal u is computed as follows: after each idle period the signal u is additively decreased by a constant $\alpha > 0$ and after each busy period the signal u is additively increased by a constant $\gamma > 0$. If u_i becomes negative then we set it equal to 0. Note that this rule follows the intuition that the congestion signal u should be increased when the channel is busy, and be decreased when the channel is idle.

One can show that the probability P_b that at least one node makes a transmission attempt after an idle slot is given by $P_b = 1 - e^{-G}$ (see for example [6]). The expected change Δu in the signal u between two idle periods of length L_i is then equal to

$$\begin{aligned} \Delta u &= -\alpha(1 - P_b) + (-\alpha + \gamma)P_b \\ &= -\alpha + \gamma P_b = -\alpha + \gamma(1 - e^{-G}). \end{aligned}$$

Let G^* be given by

$$G^* = \ln\big(\frac{\gamma}{\gamma - \alpha}\big). \tag{12}$$

Note that for $G = G^*$, the expected change in the congestion signal is equal to 0, i.e. we have

$$-\alpha + \gamma(1 - e^{-G^*}) = 0.$$

Furthermore, it can be shown that if the offered load G is smaller than G^* then the congestion signal u tends to decrease (and hence the packet-drop probability tends to decrease), whereas for $G > G^*$ the congestion signal u tends to increase (and hence the packet-drop probability tends to increase). It follows that G^* is the unique operating point and the above active queue management mechanism will stabilize the offered load at G^*.

Eq. (12) provides a simple way for setting the system throughput. Suppose that we wish to set the throughput equal to $X(G^*)$, $0 < G^* \leq G^+(\beta)$; this can be achieved by choosing $\gamma > 0$ and set α equal to

$$\alpha = \gamma(1 - e^{-G^*}). \tag{13}$$

Using an operating point analysis similar to the one given in [7], one can show that the performance at the operating point of the above distributed scheduling and active queue management algorithm satisfies Property 1

VIII. A Distributed Mechanism for Multihop Networks

In this section, we extend the above mechanism to multihop networks, where instead of an active queue management we consider a packet marking mechanism as it was proposed and analyzed by Lapsley and Low in the context of rate control in wireline networks [8].

A. Packet Marking and Scheduling

We consider the following packet marking and scheduling mechanism. Each node monitors the activities to update its congestion signal u_i. We say that node i has been idle for β time units if i hasn't sent, nor received, a packet for β time units. We refer to such a period as an idle period. The signal u_i is then updated as follows. After each idle period, node i decreases its signal u_i by a factor $\alpha > 0$. After each busy period (which could either be a successful transmission or a collision), node i increases its congestion signal u_i by a factor $\beta > 0$. More precisely, after an idle period the congestion signal gets updated to $\max\{0, u_i - \alpha\}$ and after a busy period the congestion signal gets update to $u_i + \gamma$.

Using the signal u_i, node i then marks incoming packets with probability

$$m_i = 1 - \phi^{-u_i},$$

where $\phi > 1$ is a given constant (see also [8]).

Using packet marking, scheduling is then done as follows. Consider the backlogged packets at node i. We say that a backlogged packet at node i is unmarked if it has not been marked by node i nor by any other node along its route before node i. Let $\bar{b}_{(i,j)}$ be the number of unmarked packets at node i that are to be transmitted over link (i,j), $i \in \mathcal{N}_i$. When node i senses that link (i,j) has been idle for a period of β time units, then it starts a packet transmission on link (i,j) with probability $\min\{q\bar{b}_{(i,j)}, 1-\epsilon\}$ where ϵ is a small constant. Note that the resulting transmission probabilities are proportional to the number of unmarked backlog packets for a given link (i,j).

Let G_i be the offered load at node i (expected number of transmission attempts between two idle periods of length β) and let

$$G^* = \ln\left(\frac{\gamma}{\gamma-\alpha}\right). \tag{14}$$

The same analysis as given in Section VII-D shows that under the above packet marking and scheduling mechanism node i tries to stabilize G_i ant G^*.

B. Rate Control

We can use the above packet marking mechanism to implement a rate control scheme as it was proposed and analyzed by Lapsley and Low in the context wireline networks [8].

Suppose that for each route $r \in \mathcal{R}$, the source node s_r receives from the receiving node r about the fraction of packets that were dropped along route r. The end-to-end probability that a packet along r is marked is given by

$$m_r = 1 - \prod_{j\in r}(1-m_j) = 1 - \phi^{-\sum_{j\in r} u_j}.$$

Let $u_r = \sum_{j\in r} u_r$ be the sum of the congestion signals along route r, then the source node s_r sets its packet transmission rate λ_r for route r equal to

$$\lambda_r = U_r'^{-1}(u_r)$$

and $U_r'^{-1}$ is the inverse of the derivative of the utility U_r.

C. Bottleneck Node

Using the CSMA fixed point approximation of Section V, we define a backlogged node i under the above scheduling and rate control mechanism as follows.

Let $\bar{\mathbf{b}} = \{\bar{b}_{(i,j)}\}_{(i,j)\in\mathcal{L}}$ be the current backlog of unmarked packets at links $(i,j) \in \mathcal{L}$, and suppose that we have that

$$qp_{(i,j)} \leq 1-\epsilon, \qquad (i,j) \in \mathcal{L}.$$

Then $\bar{\mathbf{b}}$ defines a CSMA policy $\mathbf{p}$ given by

$$\mathbf{p} = q\bar{\mathbf{b}}.$$

Let $G(\mathbf{p})$ be the corresponding CSMA fixed point and let $G_i(\mathbf{p})$ be the offered load at node i (expected number of transmission attempts between two idle periods of length β). Then under the CSMA fixed point approximation, the throughput $\tau_i(\mathbf{p})$ at node i is given by

$$\tau_i(\mathbf{p}) = \frac{G_i(\mathbf{p})e^{-G_i(\mathbf{p})}}{1+\beta-e^{-G_i(\mathbf{p})}} \sum_{j\in\mathcal{N}_i} \kappa_{(i,j)}(\mathbf{p})e^{-G_j(\mathbf{p})}$$

where

$$\kappa_{(i,j)}(\mathbf{p}) = \frac{(p_{(i,j)}+p_{(j,i)})\rho_j(\mathbf{p})}{G_i(\mathbf{p})}.$$

Given a vector $\bar{\mathbf{b}}$ such that

$$qp_{(i,j)} \leq 1-\epsilon, \qquad (i,j) \in \mathcal{L},$$

we say that node i is a bottleneck node if for $\mathbf{p} = q\bar{\mathbf{b}}$ we have that $G_i(\mathbf{p}^*) = G^* = \ln\left(\frac{\beta}{\beta-\alpha}\right)$.

For a given vector $\bar{\mathbf{b}}$ and CSMA policy $\mathbf{p} = q\bar{\mathbf{b}}$, suppose that node i is a bottleneck node. Then under the CSMA fixed point approximation the node throughput $\tau_i(\mathbf{p})$ is bounded by

$$\tau(G^*)e^{-G^*} \leq \tau_i(\mathbf{p}) \leq \tau(G^*)$$

where

$$\tau(G^*) = \frac{G^*e^{-G^*}}{1+\beta-e^{-G^*}}$$

where G^* is as given in Eq. (14).

D. Operating Point Analysis

In the following, we use an operating point analysis to characterize the rate allocation under the above scheduling and rate control mechanisms. Throughout this subsection we assume that G^* given by Eq. (14) is such that

$$\tau(G^*) \leq \tau(G^+(\beta))e^{-G^+(\beta)}$$

where $G^+(\beta)$ is as defined in Subsection V-A.

To do that, suppose that we are given a network $(\mathcal{L},\mathcal{R})$ and suppose that the parameters of the marking algorithm described above are equal to α and γ. Let $G^* = \ln(\gamma/(\gamma-\alpha))$ as defined above. We then characterize the system state the two vectors by $(\lambda, \bar{\mathbf{b}})$ where $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$ characterizes the rate allocation and $\bar{\mathbf{b}} = \{\bar{b}_{(i,j)}\}_{(i,j)\in\mathcal{L}}$ characterizes the backlog of unmarked packets at links $(i,j) \in \mathcal{L}$. At a state $(\lambda, \bar{\mathbf{b}})$, the CSMA policy $\mathbf{p}$ used to schedule packet transmission is then given by $p_{(i,j)} = \min\{q\bar{b}_{(i,j)}, 1-\epsilon\}$, $(i,j) \in \mathcal{L}$. Let $G(\mathbf{p})$, $i \in \mathcal{N}$ the corresponding CSMA fixed point and let then $C_i(\mathbf{p})$ be given by

$$C_i(\mathbf{p}) = \tau(G^*) \sum_{j\in\mathcal{N}_i} \kappa_{(i,j)}(\mathbf{p})e^{-G_j(\mathbf{p})}, \qquad i \in \mathcal{N},$$

where $\tau(G^*)$ and $\kappa_{(i,j)}(\mathbf{p})$ are as defined in the previous subsection.

We then say that $(\lambda, \bar{\mathbf{b}})$ is an operating point if the following two conditions hold: (1) λ is the solution to

$$\begin{aligned}
&\max. \sum_{r\in\mathcal{R}} U_r(\lambda_r) \\
&s.t. \left[\sum_{r:(i,j)\in r} \lambda_r + \sum_{r:(j,i)\in r}\right] \lambda_r \leq C_i(\mathbf{p}), \quad i \in \mathcal{N} \\
&\quad \lambda_r \geq 0, r \in \mathcal{R},
\end{aligned}$$

and (2) we have $G_i(\mathbf{p}) \leq G^*$, $i \in \mathcal{N}$.

Suppose that node i is a bottleneck node at an operating point $(\lambda, \bar{\mathbf{b}})$ where a bottleneck node is given as defined in the previous subsection. Then we have for node i that

$$\sum_{r:(i,j)\in r} \lambda_r + \sum_{r:(j,i)\in r} \lambda_r = C_i(\mathbf{p}).$$

One can show that the network is in equilibrium at an operating point $(\lambda, \bar{\mathbf{b}})$ in the sense that the following two properties hold: (1) we have that

$$\lambda_{(i,j)} = \sum_{r:(i,j)\in r} \lambda_r = \tau(\mathbf{p})_{(i,j)}$$

where $\tau_{(i,j)}(\mathbf{p})$ is the service rate for link (i,j) under the CSMA policy $\mathbf{p} = q\bar{\mathbf{b}}$ at the operating point $(\lambda, \bar{\mathbf{b}})$, and (2) the expected change of the congestion signal u_i is equal to 0 for all nodes $i \in \mathcal{N}$.

Given that the parameter ϵ that is used to define a scheduling policy $\mathbf{p}$ as a function of the vector of unmarked backlogged packets $\bar{\mathbf{b}}$, one can show that there always exists an operating point. The next proposition characterizes the rate vector λ at an operating point.

Proposition 2: Let $(\lambda, \bar{\mathbf{b}})$ be an operating point, then there exists a constant K such that

$$|\lambda_r^* - \lambda_r| \leq K e^{-G^*}, \qquad r \in \mathcal{R}$$

where the rate vector λ^* is the optimal solution to

$$\max. \sum_{r\in\mathcal{R}} U_r(\lambda_r) \tag{15}$$
$$s.t. \left[\sum_{r:(i,j)\in r} \lambda_r + \sum_{r:(j,i)\in r} \lambda_r\right] \leq X(G^*)e^{-G^*},\ i \in \mathcal{N}$$
$$\lambda_r \geq 0, r \in \mathcal{R}.$$

The above result states that the rate vector λ at an operating point will be close to the unique solution λ^* to the optimization problem given by Eq. (15). Furthermore, λ will converge to λ^* as the sensing period β becomes small.

The above result provides a performance characterization of the CSMA policy and rate control mechanism given in Section VIII-A and VIII-B using the CSMA fixed point approximation. Similar to Section V-C, this characterization is accurate in the limiting regime of a large network with many small flows as shown in the next subsection.

E. Asymptotic Accuracy

Consider a sequence of networks with N and sensing period $\beta^{(N)}$ such that $\lim_{N\to\infty} N\beta^{(N)} = 0$. Furthermore, consider a sequence of marking policies given by $\alpha^{(N)}$ and $\gamma^{(N)}$ such that for

$$G^{(N)} = \ln\Big(\frac{\gamma^{(N)}}{\gamma^{(N)} - \alpha^{(N)}}\Big)$$

we have that $\lim_{N\to\infty} = G_\infty$ and

$$\tau(G_\infty) < 1.$$

Let then $\lambda^{(N)}$ be the optimal solution to

$$\max. \sum_{r\in\mathcal{R}^{(N)}} U_r(\lambda_r) \tag{16}$$
$$s.t. \left[\sum_{r:(i,j)\in r} \lambda_r + \sum_{r:(j,i)\in r} \lambda_r\right]$$
$$\leq \tau(G_\infty)e^{-G_\infty}, \quad i \in \mathcal{N}^{(N)}$$
$$\lambda_r \geq 0, \quad r \in \mathcal{R}^{(N)},$$

and let $\mathbf{p}^{(N)}$ the corresponding CSMA policy such that

$$\lambda_{(i,j)}^{(N)} < \tau_{(i,j)}(\mathbf{p}^{(N)}),\ (i,j) \in \mathcal{L}^{(N)}, .$$

Then we have the following result.

Proposition 3: Let $\bar{\lambda}^{(N)}$ be the actual end-to-end rate vector obtained at an operating point $(\lambda^{(N)}, \bar{\mathbf{b}}^{(N)})$ for the network with N in the above scaling. If

$$\lim_{N\to\infty} \left(\max_{(i,j)} \lambda_{(i,j)}^{(N)}\right) = 0,$$

then we have that

$$\lim_{N\to\infty} \left(\max_{r\in\mathcal{R}^{\in}} \left|\frac{\bar{\lambda}_r^{(N)}}{\lambda_r^{(N)}} - 1\right|\right) = 11.$$

The above result states that if the network serves many small flows, then for large N the rate allocation under the scheduling and rate control mechanism of Section VIII-A and VIII-B is indeed accurately given by the utility maximization problem given by Eq. (16). Furthermore, the mechanism can be used to achieve arbitrarily close fairness and throughput-optimality.

We presented a distributed scheduling and rate control to achieve fairness and throughput-optimality in multihop wireless networks. Our results are based on an operating point analysis. Future work is to investigate the dynamics of the proposed mechanisms and shows that they indeed converge to an operating point.

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