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On the Throughput-Optimality of CSMA Policies in Multihop Wireless Networks

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Abstract. We analyze the class of Carrier Sense Multiple Access (CSMA) policies for scheduling packet transmissions in multihop wireless networks with primary interference constraints. Our main result is to establish that, in an appropriate limiting regime of large networks with a small sensing period, CSMA policies are capacity-achieving. While such efficiency characteristics of CSMA policies has been well-established in the special case of single-hop networks, their analysis for general multihop networks has been an open problem due to the complexity of the interaction among coupled interference constraints.

To answer this problem, we first introduce a novel fixed point equation to approximate the CSMA policy performance, and prove that the approximation becomes accurate for large networks with a small sensing period. Then, we use this approximation to characterize the achievable rate region of static CSMA schedulers, and show that the achievable rate region asymptotically converges to the capacity region of the network under primary interference constraints. This work reveals that random access schemes, which can operate asynchronously, can be used in large networks to serve many flows efficiently. Our empirical investigations also indicate that the accuracy of our fixed point formulation is achieved even for moderately sized networks.

This result has important implications for network algorithm design for ad hoc networks in that it reveals for the first time that random channel access schemes with attractive complexity, overhead, distributiveness, and asynchronism characteristics can achieve maximum throughput in large multihop networks.

Contents

1 Introduction

In this paper, we investigate Carrier Sense Multiple Access (CSMA) policies for multihop wireless networks in which nodes operate asynchronously and sense the wireless channel before making an attempt to transmit a packet, which may result in collisions. Our aim is to understand the achievable rate region of such random access policies. For fully-connected (also called single-hop) wireless networks where all nodes are within transmission range of each other, it is well-known that CSMA policies are throughput-optimal[1] in the limiting regime of networks with many small flows and small sensing periods [1]. However the analysis of CSMA schedulers for general multihop networks has been an open problem due to the complexity in the interaction amongst interfering transmissions. As a result, currently very little is known about the performance of CSMA in wireless multihop networks, and whether they are still throughput-optimal in this case.

Here, we analyze the performance of CSMA policies for general wireless networks with primary interference constraints. *The main contribution of the paper is to show that CSMA policies are asymptotically throughput-optimal for large networks with many small flows and a small sensing time.* This is a surprising and counter-intuitive result, and we provide a discussion in Section 5 to give intuition why this result is true.

The main results that we obtain in our analysis are as follows. We introduce a novel fixed point approximation, called the *CSMA fixed point*, to characterize the service rates of CSMA schedulers, and show that the fixed point approximation is asymptotically accurate for large networks with a small sensing time. Using the fixed point approximation, we characterize the achievable rate region of the static CSMA schedulers for networks with primary interference constraints. We show that the achievable rate region converges to the capacity region for networks with many small flows and a small sensing time.

This paper is a continuation of our earlier work [15], in which we analyzed the synchronized operation of CSMA policies. This assumption greatly simplifies the analysis but is not realistic for a practical network. Here, we provide the analysis and results for the general case of an asynchronous operation of the networks, and we provide numerical results to confirm our theoretical statements and to investigate the accuracy of our result for moderate sized networks.

Our results have important implications. While the design of throughput-optimal network algorithm design has attracted much attention (see, for example, [3,5,6,8,11,17,18,21,24,25] and references therein) in the last decade following the seminal work of Tassiulas and Ephremides [22], there are important issues such as computational and communication complexity, synchronization, and distributiveness that need to be resolved before their practical implementation. Our results show that CSMA policies can also achieve throughput-optimality in large multihop networks and are potentially much simpler to implement; hence, opening up the possibility of obtaining practical distributed resource allocation mechanisms for wireless multihop networks that are throughput optimal.

The paper is organized as follows. In Section 3, we define our system model and in Section 4 we describe the main components of the class of CSMA policies we consider in this paper. In Section 5 we provide a summary and discussion of our main result. We provide our fixed point formulation and prove its asymptotic accuracy in Sections 7-sec:asymp. In Section 10 and 11, we respectively provide a characterization of the achievable rate region of the class of CSMA policies, and show that it is asymptotically capacity achieving. We end with concluding remarks in Section 12.

2 Related Work

For single-hop networks where all nodes are within transmission range of each other, the performance of CSMA policies is well-understood. For the case where nodes are saturated and always have a packet to sent, the achievable rate region of CSMA policies is easily obtained [2]. For the case where nodes only make a transmission attempt when they have a packet to transmit has recently been derived for the limiting regime of many small flows [2]. Furthermore, the well-known "infinite node" approximations provides a simple characterization for the throughput of a given CSMA policy, as well as the achievable

[1]An algorithm is said to be throughput-optimal if it stabilizes the network queues for flow rates that are stabilizable by any other algorithm.

rate region of CSMA policies, in the case of a single-hop networks [1]. This approximation has been instrumental in the understanding of the performance of CSMA policies, as well as for the design of practical protocols for wireless local area networks [1].

For general multihop networks, results for CSMA policies are available for idealized situation of instantaneous channel feedback. This assumption of instantaneous channel feedback allows the elimination of collisions, which significantly simplifies the analysis. Under this assumption, Jiang and Walrand in [10] derived a dynamic CSMA policy that, combined with rate control, achieves throughput-optimality while satisfying a given fairness criterion. Similar results have been independently derived by Shah and Sreevastsa in [20] in the context of optical networks. While these results are obtained for a simplified (idealized) model of CSMA policies, they suggest the fact that CSMA policies might be throughput-optimal. In this work, we confirm this suggestion in the presence of collisions. It should be noted that the result by Jiang and Walrand has been obtained for general interference models, whereas our analysis focuses on wireless networks with primary interference constraints. However, our analysis does not assume instantaneous channel feedback and hence takes packet collisions into account.

The paper establishes a connection between the (asymptotic) behavior of CSMA in multihop wireless networks with well-known results for the Erlang fixed point in loss networks [12]). In particular, our proof for showing that the CSMA fixed point formulation is asymptotically accurate heavily uses results and techniques developed by Hajek and Krishna for the special case of a loss network where each link has capacity one, and the route of each connection consists of two links [9]. This suggests, as it was also noted in [2], that there is a close connection between the analysis of CSMA policies in multihop networks and loss networks. Moreover, our analysis suggests that for the well-known limiting regimes for which it can be shown that the Erlang fixed point is asymptotically accurate (see for example [12]), it should also be possible to show the same result for the CSMA fixed point presented in this paper. Exploring this connection beyond the limiting regime considered in this paper is future work.

3 System Model

In this section we describe the network and the traffic model assumed in our paper (see also Fig. 1).

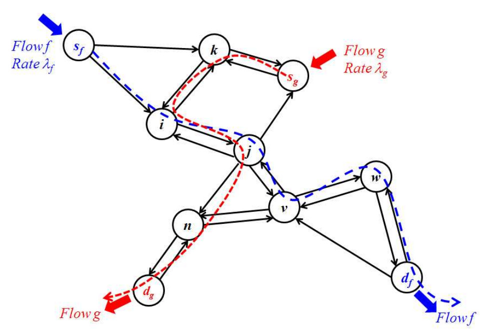

Figure 1: Example of a network where two routes f and g given by $\mathcal{R}_f = \{(s_f, i), (i, j), (j, v), (v, w), (w, d_f)\}$ and $\mathcal{R}_g = \{(s_g, k), (k, i), (i, j), (j, n), (n, d_g)\}$. In this network: the set of upstream neighbors of node j is given by $\mathcal{U}_j = \{i, v\}$; the set of downstream neighbors of node j is given by $\mathcal{D}_j = \{i, s_g, n, v\}$; the set of outgoing links of node j is given by $\mathcal{L}_j = \{(j, i), (j, s_g), (j, v), (j, n)\}$; and the set of links that interfere with (i, j) is given by $\mathcal{I}_{(i,j)} = \{(j, i), (s_f, i), (i, k), (k, i), (j, s_g), (j, v), (v, j), (j, n)\}$; the mean rate on link (i, j) is given by $\lambda_{(i,j)} = \lambda_f + \lambda_g$; and the load on node i is $\Lambda_i = 2\lambda_f + 2\lambda_g$.

Network Model: We consider a fixed wireless network composed of a set $\mathcal{N}$ of nodes with cardinality N, and a set $\mathcal{L}$ of directed links with cardinality L. A directed link $(i, j) \in \mathcal{L}$ indicates that node i is able to send data packets to node j. We assume that the rate of transmission is the same for all links and all packets are of a fixed length. Throughout the paper we rescale time such that the time it takes to transmit one packet is equal to one time unit.

For a given node $i \in \mathcal{N}$, let $\mathcal{U}_i := \{j \in \mathcal{N} : (i, j) \in \mathcal{L}\}$ be the set of upstream nodes, i.e. the set containing all nodes which can receive packets from i. Similarly, let $\mathcal{D}_i := \{j \in \mathcal{N} : (j, i) \in \mathcal{L}\}$ be set of downstream nodes, i.e. the set containing all nodes j from which i can receive packets. Collectively, we denote the set of all the neighbors of node i as $\mathcal{N}_i := \mathcal{U}_i \cup \mathcal{D}_i$. Also, we let $\mathcal{L}_i := \{(i,j) : j \in \mathcal{D}_i\}$ be the set of outgoing links from node i, i.e. the set of all links from node i to its upstream nodes $\mathcal{U}_i$.

Throughout the paper, we assume that $\mathcal{U}_i = \mathcal{D}_i$, for all $i \in \mathcal{N}$ so that we have $\mathcal{U}_i = \mathcal{D}_i = \mathcal{N}_i$, for each $i \in \mathcal{N}$. This assumption simplifies the notation as we can use a single set $\mathcal{N}_i$ to represent both $\mathcal{D}_i$ and $\mathcal{U}_i$. Our analysis can be extended to the more general case requiring only notational changes.

Henceforth, we will describe a network by the tuple $(\mathcal{N}, \mathcal{L})$.

Interference Model: We focus on networks under the well-known *primary interference*, or *node exclusive interference*, model [13, 23].

Definition 1 (Primary Interference Model)**.** *A packet transmission over link $(i,j) \in \mathcal{L}_i$ is successful if only if within the transmission duration*[2] *there exist no other activity over any other link $(m,n) \in \mathcal{L}$ which shares a node with (i,j).* $\diamond$

The primary interference model applies, for example, to wireless systems where multiple frequencies/codes are available (using FDMA or CDMA) to avoid interference, but each node has only a single transceiver and hence can only send to or receive from one other node at any time (see [19] for additional discussion).

We use the following notation to represent the primary interference constraints. For each link $l \in \mathcal{L}$, let $\mathcal{I}_l$ denote the set of links $l' \in \mathcal{L}$ that interfere with link l, i.e. the set of all links $l' \in \mathcal{L}$ that have a node in common with link l.

Traffic Model: We characterize the network traffic by a rate vector $\lambda := \{\lambda_r\}_{r \in \mathcal{R}}$ where $\mathcal{R}$ is the set of routes used by the traffic, and λ_r, $\lambda_r \geq 0$, is the mean rate in packets per unit time along route $r \in \mathcal{R}$.

For a given route $r \in \mathcal{R}$, let s_r be its source node and d_r be its destination node, and let

$$\mathcal{R}_r = \{(s_r, i), (i,j), \cdots, (v,w), (w, d_r)\} \subset \mathcal{L}$$

be the set of links traversed by the route. We allow several routes to be defined for a given source and destination pair (s,d), $s, d \in \mathcal{N}$.

Given the rate vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$, we let

$$\lambda_{(i,j)} := \sum_{r:(i,j) \in \mathcal{R}_r} \lambda_r, \qquad (i,j) \in \mathcal{L}, \tag{1}$$

be the mean packet arrival rate to link (i,j). Similarly, we let

$$\Lambda_i := \sum_{j \in \mathcal{N}_i} \left[\lambda_{(i,j)} + \lambda_{(j,i)}\right], \qquad i \in \mathcal{N}. \tag{2}$$

be the mean packet arrival rate to node $i \in \mathcal{N}$.

4 Policy Space and CSMA Policy Description

In this section, we introduce the space of scheduling policies that we are interested in, and provide the description of CSMA policies in that space. We also define the notions of stability and achievable rate region.

4.1 Scheduling Policies and Capacity Region

Consider a fixed network $(\mathcal{N}, \mathcal{L})$ with traffic vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$. A scheduling *policy* π then defines the rules that are used to schedule packet transmissions on each link $(i,j) \in \mathcal{L}$. In the following we focus on policies π that have a well-defined link service rates as a function of the rate vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$.

[2] Notice that our definition of interference model does not require a time slotted operation of the communication attempts, and hence applies to asynchronous network operation.

Definition 2 (Service Rate)**.** *: Consider a fixed network $(\mathcal{N}, \mathcal{L})$. The link service rate $\mu^{\pi}_{(i,j)}(\lambda)$, $(i,j) \in \mathcal{L}$, of policy π for the traffic vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$ is the fraction of time node i spends successfully transmitting packets on link (i,j) under π and λ, i.e. the fraction of time node i sends packets over link (i,j) that do not experience a collision.*

Let $\mathcal{P}$ be the class of all policies π that have well-defined link service rates. Note that this class contains a broad range of scheduling policies, including dynamic policies such as queue-length-based policies that are variations of the MaxWeight policy [22], as well as noncausal policies that know the future arrival of the flows.

We then define network stability as follows.

Definition 3 (Stability)**.** *For a given network $(\mathcal{N}, \mathcal{L})$, let $\mu^{\pi}(\lambda) = \{\mu^{\pi}_{(i,j)}(\lambda)\}_{(i,j) \in \mathcal{L}}$ the link service rates of policy π, $\pi \in \mathcal{P}$, for the rate vector $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$. We say that policy π stabilizes the network for λ if $\lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda)$, $(i,j) \in \mathcal{L}$.*

This commonly used stability criteria [22] requires that for each link (i,j) the link service rate $\mu^{\pi}_{(i,j)}(\lambda)$ is larger than the arrival rate $\lambda_{(i,j)}$. The capacity region of a network $(\mathcal{N}, \mathcal{L})$ is then defined as follows.

Definition 4. *(Capacity Region) For a given a network $(\mathcal{N}, \mathcal{L})$, the capacity region $\mathcal{C}$ is equal to the set of all traffic vectors $\lambda = \{\lambda_r\}_{r \in \mathcal{R}}$ such that there exists a policy $\pi \in \mathcal{P}$ that stabilizes the network for λ, i.e. we have*

$$\mathcal{C} = \{\lambda \geq 0 : \exists \pi \in \mathcal{P} \text{ with } \lambda_{(i,j)} < \mu^{\pi}_{(i,j)}(\lambda),\ \forall (i,j) \in \mathcal{L}\}.$$

4.2 CSMA Policies

In this paper we are interested in characterizing the performance of CSMA policies. In particular, we are interested in the question whether CSMA policies are throughput-optimal in the sense that for a given network $(\mathcal{N}, \mathcal{L})$ they are able to stabilize any rate vector λ in the capacity region $\mathcal{C}$. In this section, we describe the basic operation of the CSMA policies that we consider; a more detailed description of channel sensing and transmission scheduling mechanisms are given in Appendix A.

A CSMA policy is given by a transmission attempt probability vector $\mathbf{p} = (p_{(i,j)})_{(i,j) \in \mathcal{L}} \in [0,1]^L$ and a sensing period (or idle period) $\beta > 0$. The policy works as follows: each node, say i, senses the activity on its outgoing links $l \in \mathcal{L}_i$. We say that i has sensed link $(i,j) \in \mathcal{L}_i$ to be idle for a duration of an idle period β if for the duration of β time units we have that (a) node i has not sent or received a packet and (b) node i has sensed that node j have not sent or received a packet. If node i has sensed link $(i,j) \in \mathcal{L}_i$ to be idle for a duration of an idle period β, then i starts a transmission of a single packet on link (i,j) with probability $p_{(i,j)}$, independent of all other events in the network. If node i does not start a packet transmission, then link (i,j) has to remain idle for another period of β time units before i again has the chance to start a packet transmission. Thus, the epochs at which node i has the chance to transmit a packet on link (i,j) are separated by periods of length β during which link (i,j) is idle, and the probability that i starts a transmission on link (i,j) after the link has been idle for β time units is equal to $p_{(i,j)}$,

We assume that packet transmission attempts are made according to above description regardless of the availability of packets at the transmitter. In the event of a transmission decision in the absence of packets, the transmitting node transmits a *dummy* packet, which is discarded at the receiving end of the transmission (see also our discussion in Section 12).

The duration of an idle period β is again given relative to the length of a packet transmission which is assumed to take one unit time, i.e. if the length of an idle period is L_i seconds and the length of a packet transmission is L_p seconds, then we have $\beta = L_i/L_p$. For a fixed L_i, the duration of an idle period β will become small if we increase the packet lengths, and hence the packet transmission delay L_p.

Given the length of an idle period β, in the following we will sometimes refer to $\mathbf{p}$ as the CSMA policy.

4.3 Achievable Rate Region of CSMA Policies

We show in [16] that a CSMA policy $\mathbf{p}$ has a well-defined link service rate vector $\mu(\mathbf{p})$, i.e. CSMA policies are contained in the set $\mathcal{P}$. Note that for a given β, the link service rate under a CSMA policy depend only on the transmission attempt probability vector $\mathbf{p}$, but not on the arrival rates λ. The achievable rate region of CSMA policies is then given as follows.

Definition 5 (Achievable Rate Region of CSMA Policies)**.** *For a given network $(\mathcal{N}, \mathcal{L})$ and a given sensing period β, the* achievable rate region of CSMA policies *is given by the set of rate vectors $\lambda = \{\lambda_r\}_{r\in\mathcal{R}}$ for which there exists a CSMA policy $\mathbf{p}$ that stabilizes the network for λ, i.e. we have that $\lambda_{(i,j)} < \mu_{(i,j)}(\mathbf{p})$, $(i,j) \in \mathcal{L}$.*

5 Main Result

For given network $(\mathcal{N}, \mathcal{L})$, let $\Gamma(\beta)$ be the achievable flow rate region of CSMA policies when the duration of an idle period is equal to β. After providing a characterization of $\Gamma(\beta)$, in our main result in Section 11, we show that

$$\lim_{\beta\downarrow 0}\Gamma(\beta) = \{\lambda \geq \mathbf{0} : \Lambda_i < 1, \text{ for all } i \in \mathcal{N}\}$$

for networks with many small flows [3]. Since it is impossible for any policy to stabilize the network if for a node i we have that $\Lambda_i \geq 1$, this results states the CSMA policies are (asymptotically) throughput-optimal.

In fact, there is an interesting corollary to this result. It provides a simple characterization of the capacity region for networks with many small flows, i.e. the (asymptotic) capacity region of such networks is given by

$$\mathcal{C} = \{\lambda \geq \mathbf{0} : \Lambda_i < 1, \text{ for all } i \in \mathcal{N}\}.$$

The result that the achievable rate region of CSMA policies is asymptotically equal to $\mathcal{C}$ (for networks with primary interference constraints) may seem very surprising and counter-intuitive at first. And indeed, it is important to stress that our result does not state that the achievable rate region of CSMA policies is always equal to $\mathcal{C}$, but only under the conditions that (a) β becomes small and (b) the network resources are shared by many small flows. Let us briefly comment on these two conditions.

The fact that β needs to be small in order to obtain a large achievable rate region is rather intuitive; clearly if β is large (let's say close to 1) then the above result will not be true. The fact that we need the assumption of many small flows in order to obtain our result is illustrated by the following example.

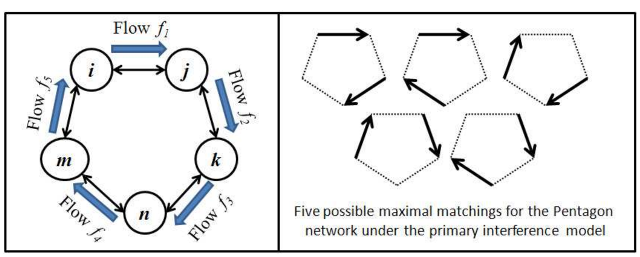

Figure 2: The pentagon network with flows $r_1, \cdots, r_5$ on each link, and the five possible simultaneous transmissions that can occur under the primary interference model. The rate $\lambda_{r_i} = (1-\epsilon)/2$, $i = 1, \cdots, 5$, for any $\epsilon \in (0, 0.1]$ is not achievable by any policy for this scenario.

Example 1. *For the pentagon network of Figure 2, let $\epsilon \in (0, 0.1]$ and $\lambda_{r_i} = (1-\epsilon)/2$ for each $r = 1, \cdots, 5$. Then, the load on each node is given by $\Lambda_i = (1-\epsilon)$ for each $i \in \mathcal{N}$. Although the resulting traffic vector λ lies within $\mathcal{C}$, no scheduling policy can stabilize the network for λ. This can be seen by noting that at most two links out of five can transmit successfully at a given time, as shown in the figure. Hence, even an optimal centralized controller cannot achieve a maximum symmetric node activity of more than $2/5$, and clearly, our result cannot hold for this network.* $\diamond$

[3] We will provide a precise description of the assumption need for the derivation of this result in Section 11.

The reason that in the pentagon network a node cannot achieve a throughput of more than 2/5 is that under each "maximal" schedule given in Figure 2, if one of the neighboring nodes of a given node i is busy transmitting, then node i has to wait for a duration of 1 time unit to get a chance to make a transmission attempt. However, if we have a network where each node i has many neighbors with which it exchanges data packets (many flows), then nodes will typically have to wait for much less than 1 time unit before it gets the chance to make start a packet transmissions. Intuitively, the larger the number of neighbors of a node, the shorter a node has to wait until it gets a chance to start a packet transmission. In addition to having many flows, we need the assumption that each flow is small in order to avoid the situation where the dynamics at each node is basically determined by a small number of large flows, essentially leading to a similar behavior as in the case where each node has only a small numbers of neighbors. Note however that these assumptions aren't sufficient in order to obtain our result; we also need to show that there exists a CSMA policy under which nodes (a) don't wait too long before making a transmission attempt (and hence waste capacity), (b) are not too aggressive such that a large fraction of packet transmissions result in collisions, and (c) share the available network resources such that the resulting link service rates support the given traffic vector $\lambda \in \mathcal{C}$.

6 Overview of Analysis

In this section, we provide a brief description of the different steps taken toward obtaining our main result that CSMA policies are (asymptotically) throughput-optimal for networks with many small flows and sufficient routing diversity.

CSMA Fixed Point Our first step is to derive a tractable way to characterize the link service rates for a given CSMA policy. Inspired by the reduced load approximations utilized in the loss network analysis [12], in Section 7.2 we propose a novel fixed point formulation to model the performance of a CSMA policy $\mathbf{p}$. Similar to the reduced load approximation in loss networks, the fixed point equation is based on an independence assumption. We show that the fixed point is well-defined, i.e. there exists a unique fixed point.

Asymptotic Accuracy of the CSMA Fixed Point We then show that the CSMA fixed point asymptotically accurate in the sense that it accurately characterizes the link service rates of a CSMA policy as β becomes small for large networks with many small flows. A technical issue the requires care in the proof is the scaling of how β becomes small and the network size N becomes large. For our proof, we use the scaling given in Assumption 1 in Section 9.

Achievable Rate Region under the CSMA Fixed Point We then use the CSMA fixed point to characterize the achievable rate region, and show that this characterization suggests that CSMA policies are throughput-optimal in the limit as the sensing time β becomes small. As part of the deriving the achievable rate region using the CSMA fixed point, we obtain an algorithm that allows to compute CSMA policy to stabilize the network for a given rate vector λ that is in the achievable rate region.

Throughput-Optimality of CSMA polices Finally, in Section 11, we derive our main result, i.e. that CSMA policies become asymptotically throughput-optimal as β becomes small for large networks with many small flows.

In our subsequent discussion, for ease of exposition we will typically refer to links as performing sensing or transmission attempts. This must be understood as the transmitting node of the (directed) link performing the action.

7 Formulation of CSMA Fixed Point Equation

In the first part of our analysis, we introduce a fixed point approximation, called the CSMA fixed point, to characterize the link service rates under a CSMA policy $\mathbf{p}$. The fixed point approximation extends the well-known infinite node approximation for single-hop networks (see for example [1]) to multihop networks which we briefly review below.

In the following we will use τ to denote the services rates obtained under our analytical formulations that we use to approximate the actual service rates μ under a CSMA policy as defined in Section 4.3.

7.1 Infinite Node Approximation for Single-Hop Networks

Consider a single-hop network where N nodes share a single communication channel, i.e. where nodes are all within transmission range of each other. In this case, a CSMA policy is given by the vector $\mathbf{p} = (p_1, \cdots, p_N) \in [0,1]^N$ where p_n is the probability that node n starts a packet transmission after an idle period of length β [1].

The network throughput, i.e. the fraction of time the channel is used to transmit packets that do not experience a collision, can then be approximated by (see for example [1])

$$\tau(G(\mathbf{p})) = \frac{G(\mathbf{p})e^{-G(\mathbf{p})}}{\beta + 1 - e^{-G(\mathbf{p})}} \tag{3}$$

where $G(\mathbf{p}) = \sum_{n=1}^{N} p_n$. Note that $G(\mathbf{p})$ captures the expected number of transmissions attempt after an idle period under a CSMA policy $\mathbf{p}$.

This well-known approximation is based on the assumption that a large (infinite) number of nodes share the communication channel. It is asymptotically accurate as the number of nodes N becomes large and each node makes a transmission attempt with a probability p_n, $n \in \mathcal{N}$ that approaches zero while the offered load $G = \sum_{n=1}^{N} p_n$ stays constant (see for example [1]).

The following results are well-known. For $\beta > 0$, one can show that

$$\tau(G) < 1, \qquad G \geq 0, \tag{4}$$

and for $G^+(\beta) = \sqrt{2\beta}$, $\beta > 0$, we have that

$$\lim_{\beta \downarrow 0} \tau(G^+(\beta)) = 1. \tag{5}$$

Using (3), the service rate $\mu_n(\mathbf{p})$ of node n under a given CSMA policy $\mathbf{p}$ can be approximated by

$$\tau_n(\mathbf{p}) \quad = \quad \frac{p_n e^{-G(\mathbf{p})}}{1 + \beta - e^{-G(\mathbf{p})}}, \qquad n = 1, ..., N. \tag{6}$$

In the above expression, p_n is the probability that node n tries to capture the channel after an idle period and $e^{-G(\mathbf{p})}$ characterizes the probability that this attempt is successful, i.e. the attempt does not collide with an attempt by any other node.

Similarly, the fraction of time that the channel is idle can be approximated by

$$\rho(\mathbf{p}) = \rho(G(\mathbf{p})) = \frac{\beta}{\beta + 1 - e^{-G(\mathbf{p})}}, \tag{7}$$

where we have that $\lim_{\beta \downarrow 0} \rho(G^+(\beta)) = 0$.

7.2 CSMA Fixed Point Approximation for Multihop Networks

We extend the above approximation for single-hop networks to multihop networks as follows.

For a given a sensing period β, we approximate the fraction of time $\rho_i(\mathbf{p})$ that node i is idle under the CSMA policy $\mathbf{p}$ by the following fixed point equation,

$$\rho_i(\mathbf{p}) = \frac{\beta}{(\beta + 1 - e^{-G_i(\mathbf{p})})}, \qquad i = 1, \cdots, N, \tag{8}$$

where

$$G_i(\mathbf{p}) = \sum_{j \in \mathcal{N}_i} \left[p_{(i,j)} + p_{(j,i)}\right] \rho_j(\mathbf{p}), \qquad i = 1, \cdots, N. \tag{9}$$

We refer to the above fixed point equation as the *CSMA fixed point* equation, and to a solution $\rho(\mathbf{p}) = (\rho_1(\mathbf{p}), \cdots, \rho_N(\mathbf{p}))$ and $G(\mathbf{p}) = (G_1(\mathbf{p}), \cdots, G_N(\mathbf{p}))$ to the fixed point equation as a *CSMA fixed point.*

The intuition behind the CSMA fixed point equation (8) and (9) is as follows: suppose that the fraction of time that node i is idle under the CSMA policy $\mathbf{p}$ is equal to $\rho_i(\mathbf{p})$, and suppose that the times when node i is idle are independent of the processes at all other nodes. If node i has been idle for β time units, i.e. node i has not received or transmitted a packet for β time units, then node i can start a transmission attempt on link (i, j), $j \in \mathcal{N}_i$, only if node j also has been idle for an idle period of β time units. Under the above independence assumption, this will be (roughly) the case with probability $\rho_j(\mathbf{p})$, and the probability that node i start a packet transmission on the link (i, j), $j \in \mathcal{N}_i$, given that it has been idle for β time units is (roughly) equal to $p_{(i,j)}\rho_j(\mathbf{p})$. Similarly, the probability that node $j \in \mathcal{N}_i$ starts a packet transmission on the link (j, i) after node i has been idle for β time units is (roughly) equal to $p_{(j,i)}\rho_j(\mathbf{p})$. Hence, the expected number of transmission attempts that node i makes or receives, after it has been idle for β time units is (roughly) given by (9). Using (7) of Section 7.1, the fraction of time that node i is idle under $\mathbf{p}$ can then be approximated by (8).

There are two important questions regarding the CSMA fixed point approximation. First, one needs to show that the CSMA fixed point is well-defined, i.e. that there always exists a unique CSMA fixed point. We have the following result.

Theorem 1. *For every CSMA policy $\mathbf{p} \in [0, 1]^L$ there exists a unique CSMA fixed point $\rho(\mathbf{p})$.*

We prove Theorem 1 in the next section.

Second, one would like to know how accurate the CSMA fixed point approximation is. In the next section, we show that the CSMA fixed point approximation is asymptotically accurate for large networks with many small flows and a small sensing period β.

For a given an sensing period β, we can then use the CSMA fixed point $G(\mathbf{p})$ for a policy $\mathbf{p}$ to approximate the link service rate $\mu_{(i,j)}(\mathbf{p})$ under a CSMA policy $\mathbf{p}$ by

$$\tau_{(i,j)}(\mathbf{p}) \quad = \quad \frac{p_{(i,j)}\rho_j(\mathbf{p})e^{-(G_i^R(\mathbf{p})+G_j(\mathbf{p}))}}{1+\beta-e^{-G_i(\mathbf{p})}} \tag{10}$$

where

$$G_i^R(\mathbf{p}) := \sum_{j \in \mathcal{N}_i} p_{(j,i)}\rho_j(\mathbf{p}).$$

Note that the above equation is similar to (6) where $p_{(i,j)}\rho_j(\mathbf{p})$ captures the probability that node i makes an attempt to capture link (i, j) if it has been idle for β time units, and $\exp\left[-(G_i^R(\mathbf{p}) + G_j(\mathbf{p}))\right]$ is the probability that this attempt is successful, i.e. the attempt does not overlap with an attempt by any other node to capture a link that has an endpoint in common with link (i, j). Note that

$$\tau_{(i,j)}(\mathbf{p}) \geq \frac{p_{(i,j)}\beta e^{-(G_i(\mathbf{p})+G_j(\mathbf{p}))}}{\left(1+\beta-e^{-G_i(\mathbf{p})}\right)\left(1+\beta-e^{-G_j(\mathbf{p})}\right)} \tag{11}$$

as $G_i(\mathbf{p}) \geq G_i^R(\mathbf{p})$.

8 Existence of a Unique CSMA Fixed Point

In this section, we prove Theorem 1 which states that there always exists a unique CSMA fixed point. We first establish the existence of a CSMA fixed point.

Proposition 1. *For every CSMA policy $\mathbf{p}$, there exists a CSMA fixed point $\rho(\mathbf{p})$.*

Proof. The proof uses the continuity properties of the fixed point equation given (8), and is a straightforward application of the Brouwer's fixed point theorem. □

We next establish the uniqueness of the CSMA fixed point for any $\mathbf{p}$. Unlike standard methods in establishing the uniqueness of a fixed point, our proof method does not require additional assumptions on the fixed point mapping, therefore may be of independent interest. The proof follows a number of steps, which is outlined here for clarity: Lemma 1 shows the uniqueness of the solution under the CSMA policy such that $p_{(i,j)} = 0$, $(i,j) \in \mathcal{L}$; Proposition 2 proves the upper-semicontinuity of the correspondence $G(\mathbf{p})$ given by (9); Proposition 3 proves that for any CSMA policy $\mathbf{p}$ and $G \in G(\mathbf{p})$, $(\mathbf{p}, G(\mathbf{p}))$ is uniquely defined in a neighborhood of $(\mathbf{p}, G(\mathbf{p}))$; finally Theorem 1 combines the preceding results to establish global uniqueness of the CSMA fixed point for any $\mathbf{p}$.

Lemma 1. *Consider CSMA policy* $\bar{\mathbf{p}}$ *with*

$$\bar{p}_{(i,j)} = 0, \quad \textit{for all } (i,j) \in \mathcal{L},$$

then for any $\beta > 0$ *the* unique *CSMA fixed point* $\rho(\bar{\mathbf{p}})$ *is given by*

$$\rho_i(\bar{\mathbf{p}}) = 1, \qquad i = 1, ..., N.$$

and

$$G_i((\bar{\mathbf{p}}) = 0, \qquad i = 1, ..., N.$$

The above results follow immediately by using the CSMA policy $\bar{\mathbf{p}}$ with $\bar{p}_{(i,j)} = 0$ in (8) and (9).

We next study the continuity properties of $G(\mathbf{p})$. The proof uses the continuity of the mapping

$$f_i(G, \mathbf{p}) = G_i - \sum_{j \in \mathcal{N}_i} \frac{\beta\big[p_{(i,j)} + p_{(j,i)}\big]}{(1 + \beta - e^{-G_j})}, \quad i = 1, \ldots, N. \tag{12}$$

Note that for $f(G, \mathbf{p}) = [f_i(G, \mathbf{p})]_{i=1,\ldots,N}$ we have that $f(G(\mathbf{p}), \mathbf{p}) = \mathbf{0}$.

Proposition 2. *The correspondence* $G : [0,1]^L \mapsto \mathbb{R}_+^N$ *is upper-semicontinuous; i.e.,* $G(\mathbf{p})$ *has a closed graph.*

Proof. Note that for all $\mathbf{p} \in [0,1]^L$, $G(\mathbf{p})$ is given by

$$G(\mathbf{p}) = \{G \in \mathbb{R}_+^N \mid f_i(G, \mathbf{p}) = 0,\ i = 1, \ldots, N\}. \tag{13}$$

We will show that G has a closed graph. Let $\{(\mathbf{p}_k, G_k)\}$ be a sequence which satisfies $G_k \in G(\mathbf{p}_k)$ for all k and converges to some $(\bar{\mathbf{p}}, \bar{G})$. Assume to arrive at a contradiction that $\bar{G} \notin G(\bar{\mathbf{p}})$. By (13), this implies that there exists some $i \in \{1, \ldots, N\}$ such that $f_i(\bar{G}, \bar{\mathbf{p}}) \neq 0$. Assume without loss of generality that there exists some $\epsilon > 0$ such that

$$f_i(\bar{G}, \bar{\mathbf{p}}) > 2\epsilon. \tag{14}$$

By the continuity of the functions f_i, we have

$$\lim_{k \to \infty} f_i(G_k, \mathbf{p}_k) = f_i(\bar{G}, \bar{\mathbf{p}}),$$

which implies the existence of some $\bar{K}$ such that

$$\Big| f_i(\bar{G}, \bar{\mathbf{p}}) - f_i(G_k, \mathbf{p}_k) \Big| \leq \epsilon, \qquad \forall\ k \geq \bar{K}.$$

Combined with (14), this yields

$$f_i(G_k, \mathbf{p}_k) \geq f_i(\bar{G}, \bar{\mathbf{p}}) - \epsilon > \epsilon,$$

contradicting the fact that $G_k \in G(\mathbf{p}_k)$ [cf. (13)]. $\square$

Recall the definition of the mapping $f(G, \mathbf{p}) = [f_i(G, \mathbf{p})]_{i=1,\ldots,N}$ given by (12). The next proposition establishes the local uniqueness of the correspondence $G(\mathbf{p})$.

Proposition 3. *For all CSMA policies* $\bar{\mathbf{p}}$ *and all CSMA fixed points* $\bar{G} \in G(\bar{\mathbf{p}})$*, there exist neighborhoods* $U \subset \mathbb{R}_+^N$ *of* $\bar{G}$ *and* $V \subset [0,1]^L$ *of* $\bar{\mathbf{p}}$ *such that for each* $\mathbf{p} \in V$ *the equation* $f(G, \mathbf{p}) = 0$ *has a unique solution* $G \in U$*. Moreover, this solution can be given by a function* $G = \phi(\mathbf{p})$ *where* ϕ *is continuously differentiable on* A*.*

Proof. We prove this statement by using the implicit function theorem.

For node $i \in \mathcal{N}$ we have

$$\frac{\partial f_i}{\partial G_j} = \begin{cases} 1 & i = j, \\ 0 & j \notin \mathcal{N}_i, \\ \psi_{(i,j)}\varphi_j & j \in \mathcal{N}_i, \end{cases}$$

with

$$\psi_{(i,j)} = \left[p_{(i,j)} + p_{(j,i)}\right] \frac{\beta}{\beta + (1 - e^{-G_j})}$$

and

$$\varphi_j = \frac{e^{-G_j}}{\beta + (1 - e^{-G_j})}.$$

Note that the function f is continuously differentiable. Therefore, in order to use the implicit function theorem we need to show that the matrix

$$\left[\frac{\partial f_i}{\partial G_j}|_{G=G(\mathbf{p})}\right] \tag{15}$$

has linearly independent rows. Before we proceed, we note that this matrix is a positive matrix.

Suppose that the rows are not linearly independent, then there exists a coefficient vector $\mathbf{x} = (x_1, ..., x_N) \neq \mathbf{0}$ such that

$$\sum_{j=1}^{N} x_j \left(\frac{\partial f_j(\mathbf{p})}{\partial G_i}\right) = 0, \qquad \text{for all } i \in \{1, \dots, N\}.$$

Using the special structure of the Jacobian matrix, we obtain

$$x_i + \varphi_i \sum_{j \in \mathcal{N}_i} \psi_{(j,i)} x_j = 0, \qquad \text{for all } i \in \{1, \dots, N\}$$

and

$$x_i = -\varphi_i \sum_{j \in \mathcal{N}_i} \psi_{(j,i)} x_j, \qquad \text{for all } i \in \{1, \dots, N\}.$$

Consider node i^* such that for all $i = 1, ..., N$ we have

$$\left|x_{i^*}[\beta + (1 - e^{-G_{i^*}})]\right| \geq \left|x_j[\beta + (1 - e^{-G_j})]\right|. \tag{16}$$

Then,

$$\begin{aligned} 1 &= -\varphi_{i^*} \sum_{j \in \mathcal{N}_{i^*}} \psi_{(j,i^*)} \frac{x_j}{x_{i^*}} \\ &\leq \varphi_{i^*} \sum_{j \in \mathcal{N}_{i^*}} \left[p_{(i^*,j)} + p_{(j,i^*)}\right] \frac{\beta}{\beta + (1 - e^{-G_{i^*}})} \cdot ... \\ &\qquad\qquad ... \cdot \frac{\left|x_j[\beta + (1 - e^{-G_j})]\right|}{\left|x_{i^*}[\beta + (1 - e^{-G_j})]\right|} \\ &\overset{(a)}{\leq} \varphi_{i^*} \sum_{j \in \mathcal{N}_{i^*}} \left[p_{(i^*,j)} + p_{(j,i^*)}\right] \frac{\beta}{\beta + (1 - e^{-G_j})} \\ &\overset{(b)}{=} \frac{G_{i^*}(\mathbf{p}) e^{-G_{i^*}(\mathbf{p})}}{\beta + (1 - e^{-G_{i^*}(\mathbf{p})})} \overset{(c)}{<} 1, \end{aligned} \tag{17}$$

where (a) follows from (16), (b) follows from the fact that $f_{i^*}(G, \mathbf{p}) = 0$, and (c) follows from (4). This proves that the Jacobian matrix in (15) is non-singular. The result follows from the implicit function theorem. ∎

We next combine Propositions 1-3 to complete the proof of Theorem 1.

of Theorem 1. By Proposition 1, for the policy $\bar{\mathbf{p}}$ with $\bar{p}_{(i,j)} = 0$, $(i,j) \in \mathcal{L}$, there exists a unique fixed point $G(\bar{\mathbf{p}})$. For a given policy $\hat{\mathbf{p}}$ define the convex combination of $\bar{\mathbf{p}}$ and $\hat{\mathbf{p}}$ as

$$\mathbf{p}(t) = (1-t)\bar{\mathbf{p}} + t\hat{\mathbf{p}}, \qquad t \in [0,1].$$

By Lemma 1, the set $G(\hat{\mathbf{p}})$ is nonempty, i.e., there exists at least one CSMA fixed point at $\hat{\mathbf{p}}$. We use the following lemma to complete the proof.

Lemma 2. *For every $\hat{G} \in G(\hat{\mathbf{p}})$, there exists a* continuous *function $\mathbf{h} : [0,1] \to \mathbb{R}_+^N$ that satisfies $\mathbf{h}(0) = G(\bar{\mathbf{p}})$, and $\mathbf{h}(1) = \hat{G}$.*

of Lemma 2. Assume, to arrive at a contradiction, that there does not exist such a continuous function. We define the set of functions

$$\mathcal{H} \triangleq \{\mathbf{h} : [0,1] \to \mathbb{R}_+^N \mid \mathbf{h}(t) \in G(\mathbf{p}(t)),\ \mathbf{h}(1) = G(\hat{\mathbf{p}})\}.$$

For all $\mathbf{h} \in H$, let $\mathcal{T}_{\mathbf{h}} \in [0,1]$ denote the set of points at which $\mathbf{h}$ is discontinuous. We define the point $\tilde{t} \in [0,1]$ as

$$\tilde{t} \triangleq \inf_{\left\{t \in \bigcup\limits_{\mathbf{h}\in\mathcal{H}} \mathcal{T}_{\mathbf{h}}\right\}} t.$$

Note that the set $\bigcup\limits_{\mathbf{h}\in\mathcal{H}} \mathcal{T}_{\mathbf{h}}$ is a bounded and closed set, therefore the minimum in the preceding optimization problem is attained, implying the existence of some $\mathbf{h} \in \mathcal{H}$ such that $\mathbf{h}$ is discontinuous at $\tilde{t}$. By the upper-semicontinuity of $G(\mathbf{p})$ (cf. Proposition 2), the function $\mathbf{h}$ can be chosen to be right continuous at $\tilde{t}$. Note also that since $G(\mathbf{p}(0))$ is unique, it follows that $\tilde{t} > 0$. By the definition of $\tilde{t}$, there exists some $\delta > 0$ such that for all $\epsilon > 0$ sufficiently small,

$$\left|\mathbf{h}(\tilde{t}) - G_\epsilon\right| > \delta, \qquad \forall\ G_\epsilon \in G(\mathbf{p}(\tilde{t} - \epsilon)).$$

This contradicts the fact that for all $\bar{\mathbf{p}}$ and $\bar{G} \in G(\bar{\mathbf{p}})$, $(\mathbf{p}, G(\mathbf{p}))$ is uniquely defined in a neighborhood of $(\bar{\mathbf{p}}, \bar{G})$ (cf. Proposition 3), proving the claim. □

Back to the Proof of Theorem 1: Assume, to arrive at a contradiction, that there exist $\hat{G}_1$ and $\hat{G}_2$ ($\hat{G}_1 \neq \hat{G}_2$) such that $\hat{G}_1,\ \hat{G}_2 \in G(\hat{\mathbf{p}})$. By Claim 2, it follows that there exist continuous functions, $\mathbf{h}_1(\cdot)$ and $\mathbf{h}_2(\cdot)$, such that $\mathbf{h}_1(0) = \mathbf{h}_2(0) = G(\bar{\mathbf{p}})$; $\mathbf{h}_1(1) = G_1(\hat{\mathbf{p}})$ and $\mathbf{h}_2(1) = G_2(\hat{\mathbf{p}})$. Then, there must exist a $\tau = \max\{t \in [0,1] : \mathbf{h}_1(t) = \mathbf{h}_2(t)\}$. Since we know that $G(\bar{\mathbf{p}})$ is unique, there must be a bifurcation of the $(\mathbf{p}(t), G(\mathbf{p}(t))$ as t exceeds τ. But, this contradicts the local *uniqueness* result of Proposition 3. Hence, $G(\hat{\mathbf{p}})$ [and therefore $\rho(\hat{\mathbf{p}})$] is unique for all $\hat{\mathbf{p}}$. □

The uniqueness result of Theorem 1 combined with the upper-semicontinuity of Proposition 2 directly implies the continuity of $G(\mathbf{p})$, and hence of $\rho(\mathbf{p})$. This is stated in the following corollary.

Corollary 1. *The fixed point $\rho(\mathbf{p})$ is continuous in $\mathbf{p}$.*

9 Asymptotic Accuracy

Consider a sequence of networks for which the number of nodes N increases to infinity. Let $\mathcal{L}^{(N)}$ be the set of all links in the network with N nodes, and let $\mathcal{N}_i^{(N)}$ be the set of neighbors of node i.

As N increases, consider a corresponding sequence of CSMA policies $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ with sensing periods $\{\beta^{(N)}\}_{N\geq 1}$, where $\mathbf{p}^{(N)}$ defines the CSMA policy for the network with N nodes. We make the following assumptions.

Assumption 1. *For the sequences $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ and $\{\beta^{(N)}\}_{N\geq 1}$ the following is true.*

(a) We have that $\lim_{N\to\infty} N\beta^{(N)} = 0$.

(b) For $p_{max}^{(N)} = \max_{(i,j)\in\mathcal{L}^{(N)}} p_{(i,j)}^{(N)}$ *we have that* $\lim_{N\to\infty} \frac{p_{max}^{(N)}}{\beta^{(N)}} = 0$.

(c) There exists a constant χ *and an integer* N_0 *such that for all* $N \geq N_0$ *we have that*

$$\sum_{j\in\mathcal{N}_i^{(N)}} [p_{(i,j)}^{(N)} + p_{(j,i)}^{(N)}] \leq \chi\beta^{(N)}, \qquad i = 1, \cdots, N.$$

These technical assumption have the following interpretation: condition (a) characterizes how fast $\beta(N)$ decreases to zero as the network size N increases, condition (b) implies that the attempt probability of each link becomes small as N becomes large, and condition (c) states that the total rate with which links that originate or end at a given node i are captured, is upper-bounded by χ.

For the above scaling, let $\rho(\mathbf{p}^{(N)}) = (\rho_1(\mathbf{p}^{(N)}), \cdots, \rho_N(\mathbf{p}^{(N)}))$ be the CSMA fixed point for the network of size N, and let $\bar{\rho}_i(\mathbf{p}^{(N)})$ be the actual fraction of time that node i is idle. Furthermore, let

$$\delta_\rho^{(N)} = \max_{i=1,\cdots,N} |\rho_i(\mathbf{p}^{(N)}) - \bar{\rho}_i(\mathbf{p}^{(N)})|$$

be the maximum approximation error of the CSMA fixed point. Similarly, let

$$\delta_\tau^{(N)} = \max_{(i,j)\in\mathcal{L}^{(N)}} \left|1 - \frac{\tau_{(i,j)}(\mathbf{p}^{(N)})}{\mu_{(i,j)}(\mathbf{p}^{(N)})}\right|$$

be the maximum relative approximation error of the link service rates under the CSMA fixed point. Note that under Assumption 1 the link service rate $\mu_{(i,j)}(\mathbf{p}^{(N)})$ will approach zero as N increases and the error term $|\tau_{(i,j)}(\mathbf{p}^{(N)}) - \mu_{(i,j)}(\mathbf{p}^{(N)})|$ will trivially vanish; this is the reason why we consider the relative error when studying the accuracy of the CSMA fixed point equation for the link service rates.

The following result states that in the limit as N approaches infinity the CSMA fixed point approximation becomes asymptotically accurate.

Theorem 2. *For the above defined scaling we have that*

$$\lim_{N\to\infty} \delta_\rho^{(N)} = 0, \qquad \textit{and} \qquad \lim_{N\to\infty} \delta_\tau^{(N)} = 0.$$

We provide a proof of Theorem 2 in Appendix B.

9.1 Numerical Results

In this section we illustrate Theorem 2 using numerical results obtain for the network given in Figure 3.

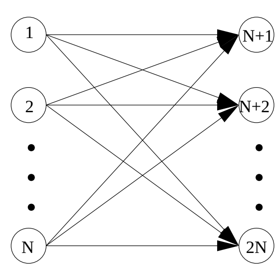

Figure 3: Network topology for our numerical results consists of a set of N sender nodes $\mathcal{N}_S = \{1, ..., N\}$, and a set of N receiver nodes $\mathcal{N}_R = \{N+1, ..., 2N\}$. The set of links $\mathcal{L}$ consists of all directed links (i, j) from a sender $i \in \mathcal{N}_S$ to a receiver $j \in \mathcal{N}_R$.

For this network, a CSMA policy $\mathbf{p} = (p_{(i,j)})_{(i,j)\in\mathcal{L}} \in [0, 1]^L$ determines the probabilities $p_{(i,j)}$ with which sender $i \in \mathcal{N}_S$ starts a transmission of a packet to receiver $j \in \mathcal{N}_R$, after link (i, j) has been sensed to be idle for sensing period of β time units. Given a sensing period β, the CSMA fixed point for a policy $\mathbf{p}$ is then given by

$$\rho_i(\mathbf{p}) = \frac{\beta}{(\beta + 1 - e^{-G_i(\mathbf{p})})}, \qquad i = 1, \cdots, 2N,$$

where

$$G_i(\mathbf{p}) = \sum_{j \in \mathcal{N}_R} p_{(i,j)} \rho_j(\mathbf{p}), \qquad i \in \mathcal{N}_S$$

and

$$G_j(\mathbf{p}) = \sum_{i \in \mathcal{N}_S} p_{(i,j)} \rho_j(\mathbf{p}), \qquad j \in \mathcal{N}_R.$$

Consider then a sequence of policies $\mathbf{p}^{(N)}$ and a sequence of sensing periods $\beta^{(N)}$ as function of size of the sender set $\mathcal{N}_S$, given by

$$\beta^{(N)} = 1/(N log(N)) \quad \text{and} \quad p_{(i,j)} = \chi \beta^{(N)}/(2N).$$

Note that this scaling satisfies Assumption 1. Furthermore, due the symmetry of the network topology as well as the CSMA policy $\mathbf{p}^{(N)}$ that we consider, the CSMA fixed point $\rho_i(\mathbf{p}^{(N)})$ is symmetric and we have that

$$\rho_i(\mathbf{p}^{(N)}) = \rho_j(\mathbf{p}^{(N)}), \qquad i, j \in \mathcal{N} = \mathcal{N}_S \cup \mathcal{N}_R.$$

We then evaluate the performance of a CSMA policy $\mathbf{p}^{(N)}$ using $\chi = 10$ for different sizes N of the sender set $\mathcal{N}_S$.

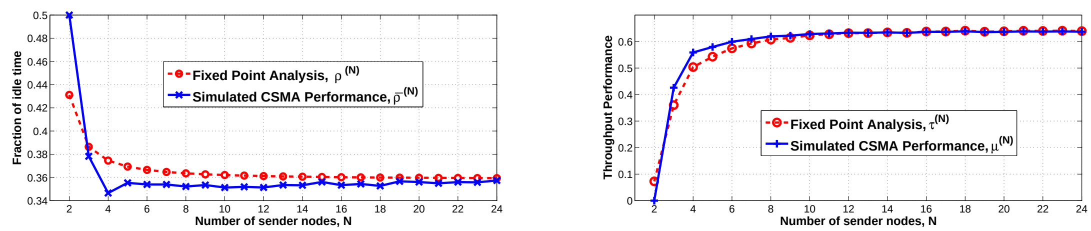

Figure 4: Comparison of the actual fraction of idle time under the CSMA policy and the predicted values based on the fixed point formulation.

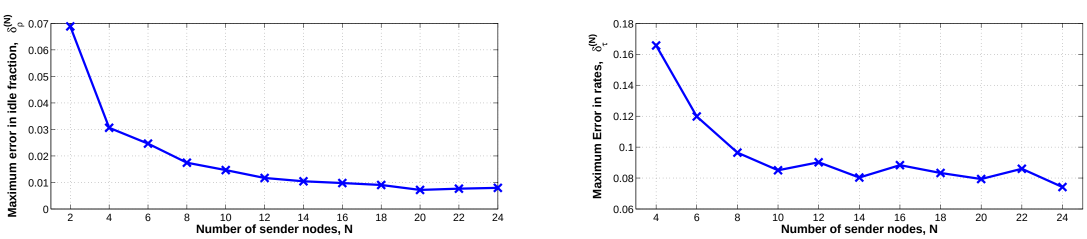

Figure 5: Error terms of Theorem 2 for different values of N.

Figure 4 shows the measured mean fraction of times that nodes are idle and mean node throughput, compared with the performance predicted by the CSMA fixed point. Figure 5 shows the error terms of Theorem 2 for the approximation error in the fraction of time that nodes are idle, and the link service rates.

Note that the above numerical results suggest that the CSMA fixed point approximation is remarkably accurate even for smaller values of N. This suggest that the CSMA fixed point approximation is not only asymptotically accurate, but is already useful to characterize the performance for networks where each nodes has a relatively small number of neighbors. However, more simulations are needed to verify this observation for more general network topologies.

10 Approximate Achievable Rate Region

In this section we use the CSMA fixed point approximation to characterize the achievable rate region of CSMA policies. In Section 11 we will show that this characterization is asymptotically accurate for large networks with a small sensing time.

Consider a network $(\mathcal{N}, \mathcal{L})$ with sensing time $\beta > 0$ as described in Section 3, let $\Gamma(\beta)$ be given by

$$\Gamma(\beta) = \left\{ \lambda = \{\lambda \geq 0 | \Lambda_i < \tau(G^+(\beta)) e^{-(G^+(\beta))}, \; i = 1, \cdots, N \right\},$$

where $G^+(\beta) = \sqrt{2\beta}$ and $\tau(G^+(\beta)$ are as defined in Section 7.1 and

$$\Lambda_i = \sum_{j \in \mathcal{N}_i} \left[\lambda_{(i,j)} + \lambda_{(j,i)} \right], \qquad i \in \mathcal{N},$$

is as defined in Section 3.

The next result states that for a network $(\mathcal{N}, \mathcal{L})$ with sensing time $\beta > 0$ the achievable rate region of CSMA policies under the CSMA fixed point approximation contains the set $\Gamma(\beta)$.

Theorem 3. *Given a network $(\mathcal{N}, \mathcal{L})$ with sensing period $\beta > 0$, for every $\lambda \in \Gamma(\beta)$ there exists a CSMA policy $\mathbf{p}$ such that*

$$\lambda_{(i,j)} < \tau_{(i,j)}(\mathbf{p}), \qquad (i,j) \in \mathcal{L}.$$

Proof. By the definition, we have $\Lambda_i < \tau(G^+(\beta)) e^{-G^+(\beta)}$ for all $i \in \mathcal{N}$.

For each node $i = 1, ..., N$, choose $G_i \in [0, G^+(\beta))$ such that

$$e^{(G_i - G^+(\beta))} \tau(G_i) e^{-G^+(\beta)} = \Lambda_i$$

and let

$$\rho_i = \frac{\beta}{\beta + 1 - e^{-G_i}}.$$

Such a G_i exists since the function

$$f(G_i) = e^{(G_i - G^+(\beta))} \tau(G_i) e^{-G^+(\beta)}$$

is continuous in G_i with $f(0) = 0$ and

$$f(G^+(\beta)) = \tau(G^+(\beta)) e^{-G^+(\beta)} > \Lambda_i(\lambda).$$

Using $\rho_i = 1, ..., N$ as defined above, consider the CSMA policy $\mathbf{p}$ given by

$$p_{(i,j)} = \frac{\lambda_{(i,j)}}{\rho_i \rho_j} \beta e^{2G^+(\beta)}, \quad (i,j) \in \mathcal{L}. \tag{18}$$

By applying the above definitions, at every node $i = 1, ..., N$ we have that

$$\begin{aligned}
\sum_{j \in \mathcal{N}_i} [p_{(i,j)} + p_{(j,i)}] \rho_j &= \sum_{j \in \mathcal{N}_i} \frac{\lambda_{(i,j)} + \lambda_{(j,i)}}{\rho_i \rho_j} \beta e^{2G^+(\beta)} \rho_j \\
&= \frac{\beta e^{2G^+(\beta)}}{\rho_i} \sum_{j \in \mathcal{N}_i} [\lambda_{(i,j)} + \lambda_{(j,i)}] = \frac{\beta e^{2G^+(\beta)}}{\rho_i} \Lambda_i(\lambda) \\
&= \frac{\beta e^{2G^+(\beta)}}{\rho_i} e^{(G_i - G^+(\beta))} \tau(G_i) e^{-G^+(\beta)} = \beta \frac{1}{\rho_i} e^{G_i} \tau(G_i) \\
&= \beta \frac{\beta + 1 - e^{-G_i}}{\beta} e^{G_i} \frac{G_i e^{-G_i}}{\beta + 1 - e^{-G_i}} = G_i.
\end{aligned}$$

This implies that the above choices of $G = (G_1, \cdots, G_N)$ and $\rho = (\rho_1, \cdots, \rho_N)$ define the CSMA fixed point of the static CSMA policy given by (18), i.e. we have that

$$\rho(\mathbf{p}) = \rho \quad \text{and} \quad G(\mathbf{p}) = G.$$

Using (11), the service rate $\tau_{(i,j)}(\mathbf{p})$ on link (i,j) under $\mathbf{p}$ is then given by

$$\begin{aligned}
\tau_{(i,j)}(\mathbf{p}) &\geq \frac{p_{(i,j)}\rho_j(\mathbf{p})e^{-(G_i(\mathbf{p})+G_j(\mathbf{p}))}}{1+\beta-e^{-G_i(\mathbf{p})}} \\
&= p_{(i,j)}\frac{\rho_j e^{-(G_i+G_j)}}{1+\beta-e^{-G_i}} = \frac{\lambda_{(i,j)}}{\rho_i\rho_j}\beta e^{2G^+(\beta)}\frac{\rho_j e^{-(G_i+G_j)}}{1+\beta-e^{-G_i}} \\
&= \lambda_{(i,j)}\frac{\beta}{\rho_i(1+\beta-e^{-G_i})}e^{2G^+(\beta)-(G_i+G_j)} \\
&= \lambda_{(i,j)}e^{2G^+(\beta)-(G_i+G_j)} > \lambda_{(i,j)},
\end{aligned}$$

where we used in the last inequality the fact that by construction we have $G_i, G_j < G^+(\beta)$. The proposition then follows. □

The proof of Theorem 3 is constructive in the sense that given a rate vector $\lambda \in \Gamma(\beta)$, we construct a CSMA policy $\mathbf{p}$ such that $\lambda_{(i,j)} < \tau_{(i,j)}(\mathbf{p})$, $(i,j) \in \mathcal{L}$. We will use this construction for our numerical results in Section 11.3.

Note that from Section 7.1, we have that

$$\lim_{\beta\to 0} G^+(\beta) = 0, \quad \text{and} \quad \lim_{\beta\to 0}\tau(G^+(\beta)) = 1.$$

Using these results, we obtain that

$$\lim_{\beta\downarrow 0}\Gamma(\beta) = \{\lambda \geq \mathbf{0} \mid \Lambda_i < 1 \ \ i = 1,\cdots,N\}.$$

Since any rate vector λ for which there exists a node i with $\Lambda_i \geq 1$ cannot be stabilized, this suggests that for network with a small sensing time the achievable rate region of static CSMA policies is equal to the capacity region, $\mathcal{C} = \{\lambda \geq \mathbf{0} \mid \Lambda_i < 1 \ \ i = 1,\cdots,N\}$.

In the next section we show that this result is true for the limiting regime of networks with many small flows.

11 Asymptotic Throughput-Optimality

In this section we derive our main result and show that CSMA policies are asymptotically throughput-optimal as the sensing period β becomes small for networks with many small flows.

11.1 Many Small Flows Asymptotic

In Section 9 we introduced a sequence of networks for which the number of nodes N increases to infinity, and let $\mathcal{L}^{(N)}$ be the set of all links in the network with N nodes, and $\mathcal{N}_i^{(N)}$ be the set of neighbors of node i in the network with N nodes. In this section, we introduce a similar scaling for the rate vectors. Let $\lambda^{(N)} = \{\lambda_r^{(N)}\}_{r\in\mathcal{R}^{(N)}}$ be the rate vector for the network with N nodes. Furthermore,

$$\lambda_{(i,j)}^{(N)} = \sum_{r\in\mathcal{R}^{(N)}:(i,j)\in r}\lambda_r^{(N)}, \qquad (i,j)\in\mathcal{L}^{(N)},$$

be the mean packet arrival rate on link (i,j), and let

$$\Lambda_i^{(N)} = \sum_{j\in\mathcal{N}_i^{(N)}}\left[\lambda_{(i,j)}^{(N)} + \lambda_{(j,i)}^{(N)}\right], \qquad i = 1,\cdots,N,$$

be the mean packet arrival rate at node i.

Definition 6 (Many Small Flows Asymptotic)**.** *Given a sequence of networks $\{\mathcal{N}^{(N)},\mathcal{L}^{(N)}\}_{N\geq 1}$, we define $\mathcal{D}$ as the set of all rate vector sequences $\{\lambda^{(N)}\}_{N\geq 1}$ such that*

$$\limsup_{N\to\infty}\left(\max_{(i,j)\in\mathcal{L}^{(N)}}\lambda_{(i,j)}^{(N)}\right) = 0.$$

The above definition characterizes the limiting regime where as the number of nodes in the network increases, we have that the mean arrival on each route becomes small, i.e. the network traffic consists of *many small flows*.

We then define the asymptotic achievable rate region of CSMA policies under the many small flows asymptotic as follows.

Definition 7. *The* asymptotic achievable rate region of static CSMA policies under the many flow limit *is the set of sequences* $\{\lambda^{(N)}\}_{N\geq 1} \in \mathcal{D}$ *for which there exists a sequence of static CSMA scheduling policies* $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ *such that*

$$\liminf_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\mu_{(i,j)}(\mathbf{p}^{(N)})}{\lambda^{(N)}_{(i,j)}} \right) > 1.$$

The above definition implies that every rate sequence $\{\lambda^{(N)}\}_{N\geq 1}$ in the asymptotic rate region can eventually be stabilized by the sequence of CSMA policy $\{\mathbf{p}^{(N)}\}_{N\geq 1}$.

Note that a sequence $\{\lambda^{(N)}\}_{N\geq 1} \in \mathcal{D}$ for which there exists a node i with

$$\lim_{N\to\infty} \Lambda_i^{(N)} \geq 1$$

can not be stabilized as service rate at each node is bounded by 1. Hence, the achievable region under the many flow limit is contained in the set

$$\mathcal{C} = \left\{ \{\lambda^{(N)}\}_{N\geq 1} \in \mathcal{D} | \ \limsup_{N\to\infty} \left(\max_{i=1,\dots,N} \Lambda_i^{(N)} \right) < 1 \right\}.$$

We refer to $\mathcal{C}$ as the capacity region under the many small flows asymptotic.

11.2 Asymptotic Rate Region

In this subsection we characterize the asymptotic achievable rate region of CSMA policies under the many small flows asymptotic for networks with a small sensing period. To do this, we again consider a sequence of sensing periods $\beta^{(N)}$, $N \geq 1$, that satisfies Assumption 1. The next theorem shows that in this case CSMA policies are throughput-optimal, i.e. the achievable rate region converges to the capacity region $\mathcal{C}$.

Theorem 4. *Given a sequence of networks* $\{\mathcal{N}^{(N)}, \mathcal{L}^{(N)}\}_{N\geq 1}$ *and a sequence of sensing periods* $(\beta^{(N)})_{N\geq 1}$ *such that*

$$\lim_{N\to\infty} N\beta^{(N)} = 0,$$

we have that for every sequence $\lambda^{(N)} \in \mathcal{C}$ *there exists a sequence of CSMA policies* $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ *that asymptotically stabilizes the network, i.e. we have*

$$\liminf_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\mu_{(i,j)}(\mathbf{p}^{(N)})}{\lambda^{(N)}_{(i,j)}} \right) > 1.$$

Proof. By definition, for each sequence $\{\lambda^{(N)}\}_{N\geq 1} \in \mathcal{C}$ there exists a scalar $\bar{\Lambda} < 1$ and an integer $\bar{N}$ such that for $N \geq \bar{N}$ we have

$$\Lambda_i^{(N)} \leq \bar{\Lambda}, \qquad i = 1, ..., N.$$

Let then Λ^* be given by

$$\Lambda^* = 1 - \frac{1-\bar{\Lambda}}{2} < 1$$

and let

$$\gamma = \frac{\Lambda^*}{\bar{\Lambda}} > 1.$$

Using this definitions, let

$$\bar{\lambda}^{(N)}_{(i,j)} = \gamma \lambda^{(N)}_{(i,j)}, \qquad (i,j) \in \mathcal{L},$$

and

$$\bar{\Lambda}^{(N)}_i = \sum_{j \in \mathcal{N}^{(N)}_i} \left[\bar{\lambda}^{(N)}_{(i,j)} + \bar{\lambda}^{(N)}_{(j,i)} \right], \qquad i \in \mathcal{N}^{(N)}.$$

For all $i \in \mathcal{N}^{(N)}$, we then have

$$\bar{\Lambda}^{(N)}_i \leq \Lambda^*, \qquad N \geq \bar{N}.$$

As $\lim_{N\to\infty} \beta^{(N)} = 0$ and $\lim_{\beta \downarrow 0} \tau(G^+(\beta)) = 1$ (see (5)), there exists a integer N_0 such that for $N \geq N_0$ we have that

$$\Lambda^* < \tau(G^+(\beta^{(N)}))e^{-(G^+(\beta^{(N)}))}, \quad i = 1, ..., N.$$

Using the proof of Theorem 3, we can then construct a sequence of CSMA policies $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ such that for $N \geq N_0$ we have

$$\bar{\lambda}^{(N)}_{(i,j)} < \tau_{(i,j)}(\mathbf{p}^{(N)}), \qquad (i,j) \in \mathcal{L}^{(N)}.$$

Using Theorem 2, the approximation $\tau_{(i,j)}(\mathbf{p}^{(N)})$ of the service rate of link (i,j) is asymptotically accurate as N increases, if the sequence $\{\mathbf{p}^{(N)}\}_{N\geq N_0}$ satisfies Assumption 1. In this case, the theorem follows as we have that

$$\begin{aligned}
&\liminf_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\mu_{(i,j)}(\mathbf{p}^{(N)})}{\lambda^{(N)}_{(i,j)}} \right) \\
&= \liminf_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\tau_{(i,j)}(\mathbf{p}^{(N)})}{\lambda^{(N)}_{(i,j)}} \frac{\mu_{(i,j)}(\mathbf{p}^{(N)})}{\tau_{(i,j)}(\mathbf{p}^{(N)})} \right) \\
&= \liminf_{N\to\infty} \left(\min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\tau_{(i,j)}(\mathbf{p}^{(N)})}{\lambda^{(N)}_{(i,j)}} \right) \\
&\geq \liminf_{N\to\infty} \left(\gamma \min_{(i,j)\in\mathcal{L}^{(N)}} \frac{\tau_{(i,j)}(\mathbf{p}^{(N)})}{\bar{\lambda}^{(N)}_{(i,j)}} \right) \\
&\geq \gamma > 1.
\end{aligned}$$

To verify Assumption 1, recall that by using the proof of Theorem 3 we obtain a sequence of CSMA policies $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ such that for $N \geq N_0$ we have

$$\bar{\lambda}^{(N)}_{(i,j)} < \tau_{(i,j)}(\mathbf{p}^{(N)}), \qquad (i,j) \in \mathcal{L}^{(N)},$$

we follows.

For a given network size N the value let $G^{(N)}_i \in [0, G^+(\beta^{(N)}))$ such that

$$e^{(G^{(N)}_i - G^+(\beta))}\tau(G^{(N)}_i)e^{-G^+(\beta)} = \bar{\Lambda}^{(N)}_i$$

and let

$$\rho^{(N)}_i = \frac{\beta^{(N)}}{\beta^{(N)} + 1 - e^{-G^{(N)}_i}}.$$

Such a $G^{(N)}_i$ exists as shown in the proof of Proposition 3.

Consider then the CSMA policy $\mathbf{p}^{(N)}$ given by

$$p^{(N)}_{(i,j)} = \frac{\bar{\lambda}^{(N)}_{(i,j)}}{\rho^{(N)}_i \rho^{(N)}_j} \beta^{(N)} e^{2G^+(\beta^{(N)})}, \quad (i,j) \in \mathcal{L};$$

using the proof of Theorem 3 we then have that

$$\bar{\lambda}^{(N)}_{(i,j)} < \tau_{(i,j)}(\mathbf{p}^{(N)}), \qquad (i,j) \in \mathcal{L}^{(N)}.$$

To verify Assumption 1 for the resulting sequence $\{\mathbf{p}^{(N)}\}_{N \geq N_0}$, we first show that for

$$p^{(N)}_{max} = \max_{(i,j) \in \mathcal{L}^{(N)}} p^{(N)}_{(i,j)}$$

we have that

$$\lim_{N \to \infty} \frac{p^{(N)}_{max}}{\beta^{(N)}} = 0.$$

To see this, note that for $\rho^{(N)}_i$ as given above, we have that

$$\rho^{(N)}_i = \frac{\beta^{(N)}}{\beta^{(N)} + 1 - e^{-G^{(N)}_i}} \geq \frac{\beta^{(N)}}{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)})}}, \quad i \in \mathcal{N}^{(N)}.$$

It then follows that

$$\frac{p^{(N)}_{(i,j)}}{\beta^{(N)}} \leq \bar{\lambda}^{(N)}_{(i,j)} \frac{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)}}}{(\beta^{(N)})^2} e^{2G^+(\beta^{(N)})}, \quad (i,j) \in \mathcal{L}.$$

Note that by definition we have

$$\lim_{N \to \infty} e^{2G^+(\beta^{(N)})} = 1.$$

Furthermore, we have that

$$\lim_{N \to \infty} \frac{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)}}}{(\beta^{(N)})^2} = \frac{4}{3}.$$

Combining the above results with the fact that for $\{\lambda^{(N)}\}_{N \geq 1} \in \mathcal{D}$ we have

$$\limsup_{N \to \infty} \left(\max_{(i,j) \in \mathcal{L}^{(N)}} \lambda^{(N)}_{(i,j)} \right) = 0,$$

it follows that

$$\lim_{N \to \infty} \frac{p^{(N)}_{max}}{\beta^{(N)}} = 0.$$

In addition, using the results that

$$\lim_{N \to \infty} \frac{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)}}}{(\beta^{(N)})^2} e^{2G^+(\beta^{(N)})} = \frac{4}{3},$$

and

$$\frac{p^{(N)}_{(i,j)}}{\beta^{(N)}} \leq \bar{\lambda}^{(N)}_{(i,j)} \frac{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)}}}{(\beta^{(N)})^2} e^{2G^+(\beta^{(N)})}, \quad (i,j) \in \mathcal{L},$$

we obtain that there exists a constant B such that for $N \geq N_0$ we have that

$$\frac{\beta^{(N)} + 1 - e^{-G^+(\beta^{(N)}}}{(\beta^{(N)})^2} e^{2G^+(\beta^{(N)})} \leq B$$

$$\sum_{j \in \mathcal{N}^{(N)}_i} \frac{[p^{(N)}_{(i,j)} + p^{(N)}_{(j,i)}]}{\beta^{(N)}} \leq B\Lambda^*, \qquad i = 1, \cdots, N.$$

Hence the sequence $\{\mathbf{p}^{(N)}\}_{N \geq N_0}$ satisfies Assumption 1 and the theorem follows. $\square$

11.3 Numerical Results

In this section, we illustrate Theorem 2 using the network topology as the one that we used for the numerical results in Section 9.1 (see also Figure 3).

As the network size increases, we consider a sequence of rate vectors idle periods $\beta^{(N)}$ and traffic vectors $\lambda^{(N)}$ given by

$$\beta^{(N)} = 0.1/(Nlog(N))$$

and

$$\lambda^{(N)}_{(i,j)} = \frac{0.95}{N} e^{-G^+(\beta^{(N)})}\tau(G^+(\beta^{(N)})), \qquad i \in \mathcal{N}_S, j \in \mathcal{N}_R.$$

We then have

$$\Lambda^{(N)}_i = \frac{0.95}{N} e^{-G^+(\beta^{(N)})}\tau(G^+(\beta^{(N)})), \qquad i \in \mathcal{N} = \mathcal{N}_S \cup \mathcal{N}_R,$$

and the rate vector is within approximate rate region $\Gamma(\beta^{(N)}$ that we obtained using CSMA fixed point approximation.

In the proof for Theorem 3, we derive the following construction for obtaining a policy $\mathbf{p}^{(N)}$ that supports a given traffic vector $\lambda \in \Gamma(\beta^{(N)})$. Let $G^{(N)} \in [0, G^+(\beta^{(N)}))$ be such that

$$e^{(G^{(N)}-G^+(\beta^{(N)}))}\tau(G^{(N)})e^{-G^+(\beta^{(N)})} = \frac{0.95}{N} e^{-G^+(\beta^{(N)})}\tau(G^+(\beta^{(N)})).$$

In the proof, we also showed that such a $G^{(N)}$ exists. Furthermore, let

$$\rho^{(N)} = \frac{\beta^{(N)}}{\beta^{(N)} + 1 - e^{-G^{(N)}}}.$$

Then the CSMA policies $\mathbf{p}^{(N)}$ with

$$p_{(i,j)} = \frac{\lambda^{(N)}}{(\rho^{(N)})^2} \beta^{(N)} e^{2G^+(\beta^{(N)})}, \quad (i,j) \in \mathcal{L}^{(N)}$$

will support the traffic vector $\lambda^{(N)}$ under the CSMA fixed point approximation.

Theorem 4 then states that for the above sequence of CSMA policies we have for a large enough N that

$$\mu^{(N)}_{(i,j)} > \lambda^{(N)}_{(i,j)}, \qquad (i,j) \in \mathcal{L}^{(N)},$$

as well as

$$\lim_{N\to\infty} \sum_{j\in\mathcal{N}_R} \mu^{(N)}_{(i,j)} > 0.95, \qquad i \in \mathcal{N}_S,$$

and

$$\lim_{N\to\infty} \sum_{i\in\mathcal{N}_S} \mu^{(N)}_{(i,j)} > 0.95, \qquad j \in \mathcal{N}_R.$$

We simulate the network to measure the true link service rates for different values of N. Figure 6 shows the average node throughput that we obtained. Note that the average node throughput indeed is above the value $\Lambda^{(N)}$ for which we designed the CSMA policy $\mathbf{p}^{(N)}$. Furthermore, as N increases the average node throughput becomes larger then 0.95 as predicted by our theoretical result. This example suggests that CSMA policy can be close to throughput optimal even if the number of neighbors of each node is relatively small.

Figure 7 shows the distribution of the ratio of link service rates to link throughputs. We know from Theorem 4 that this ratio will eventually exceed 1 for all links as N tends to infinity. We observe in Figure 7 that already at a moderate value of $N = 20$, more than 95% of the links exceed 1 and the rest of the links achieve rates close to 1. This simulation also reveals the fair nature of our CSMA policy by demonstrating that link rates are closely located.

It is interesting to contrast the measurements shown in Figure 7 with results obtained for other random access policies where a high throughput comes at the expense of severe (short) term unfairness (see for example [4]). The above constructed CSMA policy prevents such unfairness by selecting the transmission attempts carefully so that they are locally *balanced*.

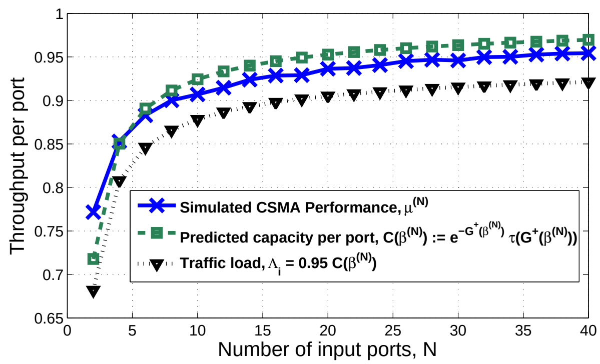

Figure 6: Performance of the CSMA policy for the network in Figure 3 with symmetric load. The policy achieves rates close to the aimed value of 0.95 per sender node even for moderate values of N.

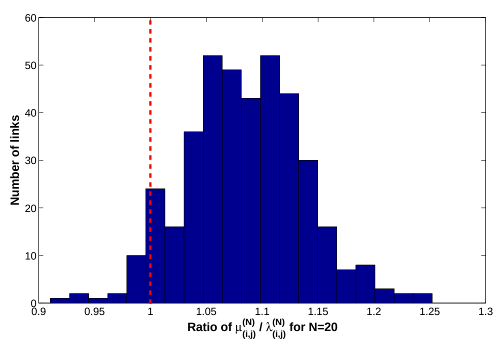

Figure 7: Distribution of the ratio of achieved rates to load on each link amongst 400 existing links in the network in Figure 3 with $N = 20$.

12 Conclusions

For our analysis, we consider CSMA policies where links make a transmission attempt with a fixed probability after the channel has been sensed to be idle, independent of the current backlog of the link. This may seem unreasonable scenario as it implies that a link might make a transmission attempt even if there is no packet to be transmitted. However, there are at least two reasons why this situation is of interest. First, such a policy could indeed be implemented (where links send dummy packets once in a while) and if we can show that it is able to achieve throughput-optimality, then it might be considered to be practical. Second, and more importantly, being able to characterize the throughput of such a policy opens up the possibility of studying more complex, dynamic CSMA policies where the attempt probabilities depend on the current backlog. In particular, the results of our analysis can be used to formulate a fluid-flow model for backlog-dependent policies, where the instantaneous throughput at a given state (backlog vector) is given by the expected throughput obtained in our analysis. Besides showing that the CSMA policies are (asymptotically) throughput-optimal, our work also provides a framework for the analysis of CSMA policies where the transmission probabilities for a given link depend on the backlog at the link. Such policies are of interest as they might allow for dynamic adaptation of the traffic load in the network. An example of such a policy, combined with rate control, is given in [14]. We are currently investigating this approach.

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A Detailed Description of CSMA Policies

In this section, we provide a detailed description of the CSMA policies that we consider.

Recall that a CSMA policy is given by a transmission attempt probability vector $\mathbf{p} = (p_{(i,j)})_{(i,j)\in\mathcal{L}} \in [0,1]^L$ and a sensing period (or idle period) $\beta > 0$. The policy works as follows: each node, say i, senses the activity on its outgoing links $l \in \mathcal{L}_i$. We say that i has sensed link $(i,j) \in \mathcal{L}_i$ to be idle for a duration of an idle period β if for the duration of β time units we have that (a) node i has not sent or received a packet and (b) node i has sensed that node j have not sent or received a packet. If node i has sensed link $(i,j) \in \mathcal{L}_i$ to be idle for a duration of an idle period β, then i starts a transmission of a single packet on link (i,j) with probability $p_{(i,j)}$, independent of all other events in the network. If node i does not start a packet transmission, then link (i,j) has to remain idle for another period of β time units before i again has the chance to start a packet transmission. Thus, the epochs at which node i has the chance to transmit a packet on link (i,j) are separated by periods of length β during which link (i,j) is idle, and the probability that i starts a transmission on link (i,j) after the link has been idle for β time units is equal to $p_{(i,j)}$,

We assume that packet transmission attempts are made according to above description regardless of the availability of packets at the transmitter. In the event of a transmission decision in the absence of packets, the transmitting node transmits a *dummy* packet, which is discarded at the receiving end of the transmission.

Below we describe more precisely the channel sensing, as well as the scheduling, mechanism of a CSMA policy.

A.1 Sensing Delay and Idle Periods

When a link l' in the interference region of a link $l \in \mathcal{L}_i$ becomes idle (or busy), then node i will be not be able to detect this instantaneously, but only after some delay, to which we refer to as the *sensing delay*.

Definition 8 (Sensing Delay $\beta_i(l)$)**.** *We let $\beta_l(l')$ denote the time it takes for node i to sense the state (active or idle) of a link l' in the interference region of a link $l \in \mathcal{L}_i$.* $\diamond$

Note that a the sensing delay a given in the above definition is lower-bounded by the propagation delay between node i and i'. The exact length of the sensing delay will depend on the specifics of the sensing mechanism deployed. We describe two possible sensing mechanisms in Appendix A.2. In the following we assume that all sensing delays are bounded by the length of an idle period β.

Assumption 2. *For all links $l \in \mathcal{L}$ we have that*

$$\beta_l(l') \leq \beta, \qquad l' \in \mathcal{I}_l.$$

In our subsequent discussion, for ease of exposition we will typically refer to links as performing sensing or transmission attempts. This must be understood as the transmitting node of the (directed) link performing the action.

A.2 Channel Sensing

The exact length of the sensing delay will depend on the specifics of the sensing mechanism deployed. Here we describe two possible approaches how channel sensing could be done for networks with primary interference constraints.

Suppose that each node $i \in \mathcal{N}$ is assigned a channel c_i over which it *receives* data packets, and suppose that the sensing radius and transmission radius of the nodes are different. The channel c_i could either be a frequency range, or a code, if a FDMA-based, or a CDMA-based, approach respectively is used to implement to obtain a network with primary interference constraints (see also our discussion in Section 3). Nodes that are within the transmission radius of a node can successfully receive its packet transmission if there are no collisions by another transmission within the transmission radius of the receiver. Nodes that are within the sensing radius of the transmitting node can only detect the presence or absence of activity together with its destination. The activity within the sensing radius does not cause collisions, but it signals the presence of activity. In this setting, a node $j \in \mathcal{N}_i$ can sense whether node i is currently sending a packet by scanning the channels c_k used by node i for transmission on its outgoing links $(i,k) \in \mathcal{L}_i$. Furthermore, if the sensing radius is at least twice the transmission radius, then a node $j \in \mathcal{N}_i$ can sense whether node i is currently receiving a packet by scanning channel c_i. Note that the time (measured in seconds) that it takes a node to detect whether a neighboring node is busy, will increase as the number of neighbors of a node increases; however, the sensing delay $\beta_l(l')$ measured relative to the time it takes to transmit a packet can still kept low by increasing the size of a packet, and hence increase the time L_p it takes to transmit a packet.

Again, suppose that each node $i \in \mathcal{N}$ is assigned a communication channel c_i over which it *receives* data packets, and that in addition it is assigned a control channel $\bar{c}_i$, where the bandwidth of the communication channel c_i is much larger than the one of the control channel $\bar{c}_i$. Then, if node i is currently receiving a packet transmission on its communication channel c_i, then it can send out a busy signal on the control channel $\bar{c}_i$. In this setting, a node $j \in \mathcal{N}_i$ can sense whether node i is currently sending a packet by scanning the channels c_k used by node i for transmission on its outgoing links $(i,k) \in \mathcal{L}_i$. Furthermore, a node $j \in \mathcal{N}_i$ can sense whether node i is currently receiving a packet by scanning the control channel $\bar{c}_i$. Again, the time (measured in seconds) that it takes a node to detect whether a neighboring node is busy, will increase as the number of neighbors of a node increases; but the sensing delay $\beta_l(l')$ measured relative to the time it takes to transmit a packet can still kept low by increasing the size of a packet. Figure 8 gives a timing-diagram for this case.

A.3 Scheduling

One additional issue that we have to account for is the event that the idle periods of two links l and l' that both originate at node i end at the same time. To prevent the possibility that node i starts

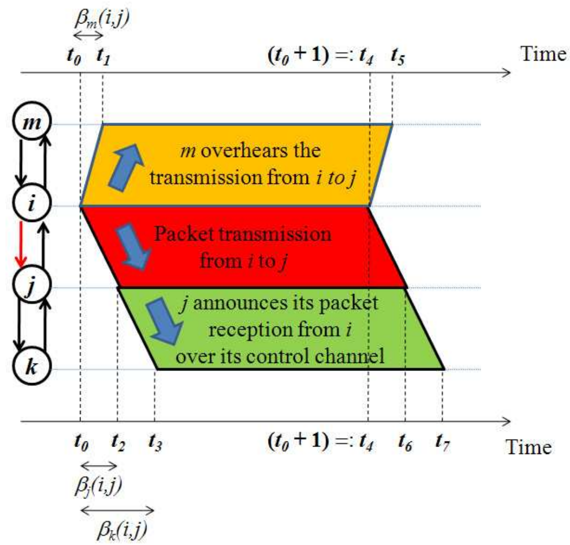

Figure 8: Nodes m, i, j, and k are connected as shown on the left. Node i starts a packet transmission to node j at t_0, which is overheard starting at t_1 by node m. Thus, the sensing delay $\beta_m(i,j)$ is equal to $(t_1 - t_0)$. Node j starts reception of the packet at t_2 (hence its sensing delay satisfies $\beta_j(i,j) = (t_2 - t_0)$) and generates a signal over its control channel$\bar{c}_j$ to indicate the activity of link (i,j). Node k senses the control signal of node j at time t_3 (hence its sensing delay is $\beta_k(i,j) = (t_3 - t_0)$. The transmission of the packet ends at time t_4 which equals $(t_0 + 1)$ since the packet transmission duration is normalized to one. Nodes m, j, and k sense the end of the activity at t_5, t_6, and t_7, respectively.

in this case a transmission on both links l and l' simultaneously (leading to sure collision), we use the following mechanism.

At a given time $t \geq 0$, let $\mathcal{L}_i(t)$ be the set of links in $\mathcal{L}_i$ for which an idle period ends at time t. If link $l = (i,j)$ belongs to the set $\mathcal{L}_i(t)$ then the probability that node i starts a transmission on link l at time t is given by

$$\frac{p_{(i,j)}}{\displaystyle\sum_{\{j' : (i,j') \in \mathcal{L}_i(t)\}} p_{(i,j')}},$$

independently of all other attempts by any node in the network.

B Proof of Theorem 2

Recall the setting for Theorem 2. We consider a sequence of networks for which the number of nodes N increases to infinity. Let $\mathcal{L}^{(N)}$ be the set of all links in the network with N nodes, and let $\mathcal{N}_i^{(N)}$ be the set of neighbours of node i. Furthermore, let $\{\mathbf{p}^{(N)}\}_{N\geq 1}$ be a sequence of CSMA policies where $\mathbf{p}^{(N)}$ defines a CSMA policies for the network with N nodes, and let $\{\beta^{(N)}\}_{N\geq 1}$ be the corresponding sequence of sensing periods. By Assumption 1, the following conditions hold.

(a) For the sequence $\{\beta^{(N)}\}_{N\geq 1}$ we have

$$\lim_{N\to\infty} N\beta^{(N)} = 0.$$

(b) For $p_{max}^{(N)} = \max\limits_{(i,j)\in\mathcal{L}^{(N)}} p_{(i,j)}^{(N)}$ we have that

$$\lim_{N\to\infty} \frac{p_{max}^{(N)}}{\beta^{(N)}} = 0.$$

(c) There exists a constant χ and an integer N_0 such that for all $N \geq N_0$ we have that

$$\sum_{j \in \mathcal{N}^{(N)}} [p_{(i,j)}^{(N)} + p_{(j,i)}^{(N)}] \leq \chi \beta^{(N)}, \qquad i = 1, ..., N.$$

For this setup, Theorem 2 states that

$$\lim_{N \to \infty} \delta_{\rho}^{(N)} = 0, \qquad \text{and} \qquad \lim_{N \to \infty} \delta_{\tau}^{(N)} = 0,$$

where $\delta_{\rho}^{(N)}$ and $\delta_{\rho}^{(N)}$ are as defined in Section 9.

To prove Theorem 2, we uses techniques and results presented by Hajek and Krishna in [9] for their analysis of blocking probabilities in loss networks. Before we start we the analysis, we provide in the next section a brief summary of the work by Hajek and Krishna in [9]. In Section B.2, we provide an overview of the proof.

B.1 Result by Hajek and Krishna

Here we provide a brief summary of the work by Hajek and Krishna, we refer to [9] for a more detailed description.

Consider a wired (loss) network consisting of a set of undirected links $\mathcal{L}$, where each link $i \in \mathcal{L}$ has capacity 1. The network serves connections (calls) where each connection uses 1 unit of the capacity at each link it traverses, i.e. each link can accommodate at most 1 connection. Furthermore, suppose that all connections use routes that consist of exactly two links. Connection requests arrive according to independent Poisson processes where $\nu_{ij} = \nu_{ji}$ denotes the arrival rate for connections that use link i and j. Once a connection is accepted, it stays in the system for an amount of time that is exponentially distributed time with mean one. If a new connection using links i and j as a route, and one these links is already serving another connection, then the newly arrived connection is blocked and lost. The Erlang fixed point equation for this loss network is then given by (see also [9])

$$\frac{B_i}{1 - B_i} = \sum_{j \in \mathcal{L}} \nu_{ij}(1 - B_j), \qquad i \in \mathcal{N}, \tag{19}$$

where B_i approximates the probability that link i is busy, i.e. serving a connection. In [9], Hajek and Krishna obtain the following result.

Proposition 4. *Consider a loss network as defined above and let*

$$r_v = \max_{i,j \in \mathcal{L}} \nu_{ij}$$

and

$$\chi = \max_{i \in \mathcal{L}} \sum_{j \in \mathcal{L}} \nu_{ij}.$$

For the actual steady-state probability $\bar{B}_i$, $i \in \mathcal{L}$, that link i is busy we have that

$$(1 - \hat{B}_i)e^{-\chi(r_v + r_v^2/2)} \leq 1 - \bar{B}_i \leq (1 - \hat{B}_i)e^{\chi(r_v + r_v^2/2)},$$

where $\hat{B}_i$, $i \in \mathcal{L}$, is the solution to the Erlang fixed point equation given by Eq. (19).

The above proposition implies that for small χ and r_v the solution to the Erlang fixed point equation approximates well the actual steady-state probability of a link being busy.

B.2 Overview of the Proof for Theorem 2

To prove Theorem 2, we will proceed as follows.

1. We relate the CSMA fixed point to the Erlang fixed point given by Eq. (19). To do this, we derive and analyze in Section B.3 and B.4 an alternative formulation of the CSMA fixed point.

2. We show that the steady-steady probabilities of nodes being idle under CSMA policy $\mathbf{p}$ are asymptotically independent. More precisely, we will first derive in Section B.6 and B.7 several properties for the the steady-steady probabilities of nodes being idle under a CSMA policy $\mathbf{p}$, which we then use to show in Section B.8 that the steady-state probabilities of nodes being idle become asymptotically independent.

3. Combining the results from step 1 and 2, we show in Section B.9 that that under Assumption 1 the solution to the CSMA fixed point equation is asymptotically accurate. In particular, we derive the following result.

   Proposition 5. *Consider a CSMA policy $\mathbf{p}^{(N)}$ for a wireless network consisting of N nodes and let*
   $$p_{max}^{(N)} = \max_{(i,j)\in\mathcal{L}} p_{(i,j)}^{(N)}$$
   and let χ be as defined in Assumption 1. Then there exist constants κ and κ_s that do not depend on N, such that the actual steady-state probabilities $\bar{\rho}_i(\mathbf{p}^{(N)})$, $i \in \mathcal{N}$, that node i is idle under the CSMA policy $\mathbf{p}^{(N)}$ satisfy
   $$\begin{aligned} \rho_i(\mathbf{p}^{(N)}) e^{-\chi(r+r^2/2)} e^{-\chi(\kappa\beta+(\kappa\beta)^2/2)} &\leq \bar{\rho}_i(\mathbf{p}^{(N)}) \\ &\leq \rho_i(\mathbf{p}^{(N)}) e^{\chi(r+r^2/2)} e^{\chi(\kappa\beta+(\kappa\beta)^2/2)}, \end{aligned}$$
   where $\rho_i(\mathbf{p}^{(N)})$ is the solution to the CSMA fixed point equation for $\mathbf{p}^{(N)}$, and
   $$r = (2N+1)(\kappa_s \beta^{(N)}) + 2r_p$$
   with
   $$r_p = \frac{p_{max}^{(N)}}{\beta^{(N)}}.$$

Using Proposition 5, we can prove Theorem 2 as follows.

Proof of Theorem 2: Consider a sequence of CSMA policies $\mathbf{p}^{(N)}$ that satisfies Assumption 1. To keep the notation light, we use in the following only $\mathbf{p}$ instead of $\mathbf{p}^{(N)}$, p_{ij} instead of $p_{(i,j)}^{(N)}$, $\bar{\rho}_i$ instead of $\bar{\rho}_i(\mathbf{p}^{(N)})$, ρ_i instead of $\rho_i(\mathbf{p}^{(N)})$, and β instead of $\beta^{(N)}$. Furthermore, we use $\mu_{(i,j)}$ instead of $\mu_{(i,j)}(\mathbf{p}^{(N)})$ to denote the link service rate for link (i,j) under the CSMA policy $\mathbf{p}^{(N)}$, and $\tau_{(i,j)}$ instead of $\tau_{(i,j)}(\mathbf{p}^{(N)})$ to denote the approximation of link service rate for link (i,j) under the CSMA fixed point approximation for the CSMA policy $\mathbf{p}^{(N)}$.

We first show that
$$\lim_{N\to\infty} \delta_\rho^{(N)} = 0.$$
This result follows immediately from Proposition 5 which states that the steady-state probabilities asymptotically converge to the solution of the CSMA fixed point equation if
$$\lim_{N\to\infty} \big((2N+1)(\kappa_s\beta) + 2r_p\big) = 0,$$
or
$$\lim_{N\to\infty} N\beta = 0$$
and
$$\lim_{N\to\infty} \frac{p_{max}^{(N)}}{\beta} = 0.$$
And indeed, these conditions hold by Assumption 1.

The proof that
$$\lim_{N\to\infty} \delta_\tau^{(N)} = 0$$
requires results that we obtain in Section B.7 and B.8; we will provide references to these results in the derivations below.

We are going to use the following convention. We say that a node i is idle if node i is currently neither sending, nor receiving, a data packet. We say that a link $l = (i, j)$ is idle if both node i and j are idle. Otherwise, we say that node i (link (i, j)) is busy.

Let y_i be the indicator whether node i is idle ($y_i = 0$) or busy ($y_i = 1$), and let $P(y_i = 0, y_j = 0)$ be the steady-state probabilities that node i and j are jointly idle. In Section B.6, we show this steady-state probability exists. Using the same argument as we give in Section B.7 to prove Lemma 20, one can show that

$$P(y_i = 0, y_j = 0)p_{(i,j)}(1 - 4\chi\beta)^2 \leq \mu_{(i,j)}\beta \leq P(y_i = 0, y_j = 0)p_{(i,j)}\frac{1}{1 - 4\chi\beta},$$

where $(1 - 4\chi\beta)$ is a lower-bounded (see Section B.7) on the probability that a packet transmission on link (i, j) is successful, i.e. does not experience a collision.

By Proposition 8 in Section B.8, we have that

$$\frac{1}{1 + 2r_p}\left(\frac{1}{1 + \kappa_s\beta}\right)^{2N} \leq \frac{P(y_i = 0, y_j = 0)}{\bar{\rho}_i\bar{\rho}_j} \leq (1 + \kappa_s\beta)^{2N}(1 + 2r_p),$$

and it follows that

$$\begin{aligned}
&\frac{1}{1 + 2r_p}\left(\frac{1}{1 + \kappa_s\beta}\right)^{2N} \bar{\rho}_i\bar{\rho}_j p_{(i,j)}(1 - 4\chi\beta)^2 \\
&\quad\leq \mu_{(i,j)}\beta \\
&\quad\leq (1 + \kappa_s\beta)^{2N}(1 + 2r_p)\bar{\rho}_i\bar{\rho}_j p_{(i,j)}\frac{1}{1 - 4\chi\beta}.
\end{aligned}$$

Combining this result with Proposition 5, we obtain that

$$\begin{aligned}
&\frac{1}{1 + 2r_p}\left(\frac{1}{1 + \kappa_s\beta}\right)^{2N} e^{-2\chi(r + r^2/2)}e^{-2\chi(\kappa\beta + (\kappa\beta)^2/2)}\rho_i\rho_j p_{(i,j)}(1 - 4\chi\beta)^2 \\
&\quad\leq \mu_{(i,j)}\beta \\
&\quad\leq (1 + \kappa_s\beta)^{2N}(1 + 2r_p)e^{2\chi(r + r^2/2)}e^{2\chi(\kappa\beta + (\kappa\beta)^2/2)}\rho_i\rho_j p_{(i,j)}\frac{1}{1 - 4\chi\beta},
\end{aligned}$$

where ρ_i and ρ_j are the solutions to the CSMA fixed point equation for the CSMA policy $\mathbf{p}$.

As we have that (see Section 7.2)

$$\frac{\rho_j p_{(i,j)}e^{-2\chi\beta}}{1 + \beta - e^{-(G_i(\mathbf{p})}} \leq \frac{\rho_j p_{(i,j)}e^{-(G_i(\mathbf{p}) + G_j(\mathbf{p}))}}{1 + \beta - e^{-G_i(\mathbf{p})}} \leq \tau_{(i,j)} \leq \frac{\rho_j p_{(i,j)}}{1 + \beta - e^{-G_i(\mathbf{p})}}$$

or

$$\frac{\rho_i\rho_j p_{(i,j)}e^{-2\chi\beta}}{\beta} \leq \tau_{(i,j)} \leq \frac{\rho_i\rho_j p_{(i,j)}}{\beta},$$

it follows that

$$\begin{aligned}
&\frac{1}{1 + 2r_p}\left(\frac{1}{1 + \kappa_s\beta}\right)^{2N} e^{-2\chi(r + r^2/2)}e^{-2\chi(\kappa\beta + (\kappa\beta)^2/2)}(1 - 4\chi\beta)e^{-2\chi\beta} \\
&\qquad\leq \frac{\tau_{(i,j)}}{\mu_{(i,j)}} \leq (1 + \kappa_s\beta)^{2N}(1 + 2r_p)e^{2\chi(r + r^2/2)}e^{2\chi(\kappa\beta + (\kappa\beta)^2/2)}\frac{1}{(1 - 4\chi\beta)^2}.
\end{aligned}$$

Note that under Assumption 1, we have that

$$\lim_{N\to\infty}\frac{1}{1 + 2r_p}\left(\frac{1}{1 + \kappa_s\beta}\right)^{2N} e^{-2\chi(r + r^2/2)}e^{-2\chi(\kappa\beta + (\kappa\beta)^2/2)}(1 - 4\chi\beta)e^{-2\chi\beta} = 1$$

and

$$\lim_{N\to\infty}(1 + \kappa_s\beta)^{2N}(1 + 2r_p)e^{2\chi(r + r^2/2)}e^{2\chi(\kappa\beta + (\kappa\beta)^2/2)}\frac{1}{1 - 4\chi\beta} = 1,$$

and it follows that

$$\lim_{N\to\infty}\delta_\tau^{(N)} = 0.$$

B.3 Alternative Formulation of the CSMA Fixed Point

In this section, we derive an alternative formulation for the CSMA fixed point for a CSMA policy $\mathbf{p}$, which we will use to related the CSMA fixed point to the Erlang fixed point for loss networks. To keep the notation light, we use in the following $p_{(i,j)}$ instead of $p^{(N)}_{(i,j}$, β instead of $\beta^{(N)}$, G_i instead of $G^{(N)}_i$, and $\mathcal{N}_i$ instead of $\mathcal{N}^{(N)}_i$.

Recall that for a CSMA policy $\mathbf{p}$ with sensing period β, the CSMA fixed point equation is given by

$$\rho_i = \frac{\beta}{\beta + 1 - e^{-G_i}}, \qquad i = 1, ..., N,$$

where

$$G_i = \sum_{j \in \mathcal{N}_i} (p_{(i,j)} + p_{(j,i)})\rho_j.$$

First we observe that for large N the offered load G_i becomes small at all nodes i.

Lemma 3. *We have that*

$$\lim_{N\to\infty} G_i = 0, \qquad i = 1, ..., N.$$

Proof. By Assumption 1, we have that

$$\lim_{N\to\infty} G_i = \lim_{N\to\infty} \sum_{j\in\mathcal{N}_i} (p_{(i,j)} + p_{(j,i)})\rho_j \le \lim_{N\to\infty} \sum_{j\in\mathcal{N}_i} (p_{(i,j)} + p_{(j,i)}) \le \lim_{N\to\infty} \chi\beta = 0.$$

□

Let

$$B_i = 1 - \rho_i.$$

The corresponding fixed point equation is given by

$$B_i = 1 - \frac{\beta}{\beta + 1 - e^{-G_i}}, \qquad i = 1, ..., N, \tag{20}$$

where

$$G_i = \sum_{j\in\mathcal{N}_i} (p_{(i,j)} + p_{(j,i)})(1 - B_j). \tag{21}$$

Note that we can rewrite the expression for B_i as

$$B_i = \frac{\beta}{\beta + 1 - e^{-G_i}} \frac{1}{\beta}(1 - e^{-G_i}) = \rho_i \frac{1}{\beta}(1 - e^{-G_i}) = (1 - B_i)\frac{1}{\beta}(1 - e^{-G_i}).$$

We then have the following result.

Lemma 4. *Given a CSMA policy* $\mathbf{p}$ *for a network with* N *nodes, let*

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}, \qquad i, j = 1, ..., N$$

and let χ *be given as in Assumption 1. Let* B_i, $i = 1, ..., N$, *be the CSMA fixed point for* $\mathbf{p}$ *as given by Eq.* (20) *and* (21)*. Then for* $\kappa \ge (e - 1)^{-1}\chi$ *we have*

$$B_i = \sum_{j\in\mathcal{N}_i} \hat{\nu}_{ij}(1 - B_i)(1 - B_j), \qquad i \in \mathcal{N},$$

where $\hat{\nu}_{ij} \ge 0$ *is such that*

$$\frac{1}{1 + \kappa\beta} \le \frac{\hat{\nu}_{ij}}{\nu_{ij}} \le 1 + \kappa\beta, \qquad (i, j) \in \mathcal{L},$$

and $\hat{\nu}_{i,j} = 0$ *if* $(i, j) \notin \mathcal{L}$.

The above lemma states that the CSMA fixed point B_i, $i = 1, ..., N$, as given by Eq. (20) and (21) can be expressed as a solution to the fixed point equation

$$B_i = \sum_{j \in \mathcal{N}_i} \hat{\nu}_{ij}(1 - B_i)(1 - B_j), \qquad i \in \mathcal{N},$$

where the true transmission rates ν_{ij} are replaced by "approximate transmission rates" $\hat{\nu}_{ij}$. Furthermore, we note that the above fixed point equation is the same as the Erlang fixed point equation given by Eq. (19), except that we allow that $\hat{\nu}_{ij} \neq \hat{\nu}_{ji}$. We use this fact in the following to related the CSMA fixed point to the Erlang fixed point considered by Hajek and Krishna.

Proof. For $G_i \in [0, 1]$, $i \in \mathcal{N}$, we have

$$G_i(1 - e^{-1}G_i) \leq 1 - e^{-G_i} \leq G_i(1 + e^{-1}G_i).$$

Furthermore, for $\kappa' \geq (e - 1)^{-1}$ we obtain that

$$\frac{G_i}{1 + \kappa' G_i} \leq 1 - e^{-G_i} \leq G_i(1 + \kappa' G_i), \qquad G_i \in [0, 1].$$

As by Assumption 1 we have that $G_i \leq \chi\beta$, it follows that for

$$\kappa \geq (e - 1)^{-1}\chi,$$

we obtain

$$\frac{G_i}{1 + \kappa\beta} \leq 1 - e^{-G_i} \leq G_i(1 + \kappa\beta)), \qquad G_i \in [0, 1].$$

Combining the above results, for $\kappa \geq (e - 1)^{-1}\chi$ we have that

$$\frac{1}{1 + \kappa\beta} \sum_{j \in \mathcal{N}_i} \nu_{ij}(1 - B_i)(1 - B_j) \leq B_i \leq (1 + \kappa\beta) \sum_{j \in \mathcal{N}_i} \nu_{ij}(1 - B_i)(1 - B_j),$$

where

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}.$$

The result then follows. $\square$

B.4 Existence and Uniqueness of a Fixed Point

Consider the fixed point equation of Lemma 4 that is given by

$$B_i = \sum_{j \in \mathcal{N}_i} \hat{\nu}_{ij}(1 - B_i)(1 - B_j), \qquad i \in \mathcal{N},$$

with

$$\hat{\nu}_{ij} \geq 0, \qquad (i, j) \in \mathcal{L},$$

where we allow that

$$\hat{\nu}_{ij} \neq \hat{\nu}_{ji}.$$

In this section, we will show that there exists a unique fixed point by using an argument that is similar to the one in Section 8 that we used to prove the existence and uniqueness of the CSMA fixed point.

We first rewrite the above fixed point equation as

$$\frac{B_i}{1 - B_i} = \sum_{j \in \mathcal{N}_i} \hat{\nu}_{ij}(1 - B_j), \qquad i \in \mathcal{N}, \tag{22}$$

where $\hat{\nu}_{ij} \geq 0$, $(i, j) \in \mathcal{L}$.

Given vector $\hat{\nu} = (\hat{\nu}_{ij})_{(i,j) \in \mathcal{L}}$ with $\hat{\nu}_{ij} \geq 0$, $(i, j) \in \mathcal{L}$, let $B(\hat{\nu})$ be the set of fixed points for Eq. (22). Then we have the following result.

Lemma 5. *For all fixed points $\bar{B} \in B(\hat{\nu})$, there exist neighbourhoods $U \subset \mathbb{R}_+^N$ of $\bar{B}$ and $V \subset [0,1]^L$ of $\hat{\nu}$ such that for each $\nu \in V$ the equation $F(B,\nu) = (F_i(B,\nu))_{i\in\mathcal{N}} = 0$ where*

$$F_i(B,\nu) = \frac{B_i}{1-B_i} - \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j),$$

has a unique solution $B \in U$. Moreover, this solution can be given by a function $B = \phi(\nu)$ where ϕ is continuously differentiable on V.

Proof. For $i \in \mathcal{N}$, we have

$$\frac{\partial F_i}{\partial B_j} = \begin{cases} \frac{1}{(1-B_i)^2}, & i = j, \\ \nu_{ij}, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases}$$

Note that the function F is continuously differentiable. Next we show that the Jacobain matrix

$$\left[\frac{\partial F_i}{\partial B_j}|_{G=G(\nu)}\right]$$

has linearly independent rows. Having established this result, the lemma then follows from the implicit function theorem. Before we proceed, we note that this matrix has non-negative entries.

Suppose that the rows are not linearly independent, then there exists a coefficient vector $\mathbf{x} = (x_1, ..., x_N) \neq \mathbf{0}$ such that

$$\sum_{j=1}^{N} x_j \left(\frac{\partial F_i(\mathbf{p})}{\partial B_j}\right) = 0, \qquad \text{for all } i \in \{1,\ldots,N\}.$$

Using the special structure of the Jacobian matrix, we obtain

$$\frac{x_i}{(1-B_i)^2} + \sum_{j\in\mathcal{N}_i} x_j \nu_{ij} = 0, \qquad i \in \mathcal{N},$$

or

$$1 + \sum_{j\in\mathcal{N}_i} \nu_{ij} \frac{x_j}{x_i}(1-B_i)^2 = 0, \qquad i \in \mathcal{N}.$$

Consider a node i^* such that

$$\left|\frac{x_{i^*}}{1-B_{i^*}}\right| \geq \left|\frac{x_i}{1-B_i}\right|, \qquad i \in \mathcal{N}.$$

Then,

$$\begin{aligned} 1 &= -\sum_{j\in\mathcal{N}_{i^*}} \nu_{i^*j}(1-B_{i^*})(1-B_j)\frac{x_j}{x_{i^*}}\frac{1-B_{i^*}}{1-B_j} \\ &\leq \sum_{j\in\mathcal{N}_{i^*}} \nu_{i^*j}(1-B_{i^*})(1-B_j)\left|\frac{x_j}{x_{i^*}}\frac{1-B_{i^*}}{1-B_j}\right| \\ &\leq \sum_{j\in\mathcal{N}_{i^*}} \nu_{i^*j}(1-B_{i^*})(1-B_j) = B_{i^*} < 1. \end{aligned}$$

Hence, we obtain a contradiction and the result follows. □

The following result we then obtain by the same argument as given for the uniqueness of the CSMA fixed point.

Lemma 6. *There exists a unique fixed point to Eq. (22).*

B.5 Sensitivity Analysis

In this section we show that asymptotically (as N becomes large) the solution to the CSMA fixed point equation converges to the solution of the Erlang fixed point equation given by Eq. (19). To show this, we use the sensitivity analysis as given by Hajek and Krishna in Section 4 of [9] with only minor notational changes. For convenience, we provide below the analysis of Hajek and Krishna applied to our setting.

Given vector $\nu = (\nu_{ij})_{(i,j)\in\mathcal{L}}$ with $\nu_{ij} \geq 0$, $(i,j) \in \mathcal{L}$, let $B = (B_1, ..., B_N)$ be the the fixed point to the equation

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i \in \mathcal{N}, \tag{23}$$

where we allow that $\nu_{ij} \neq \nu_{ji}$. Furthermore, let the links $l = (i,j) \in \mathcal{L}$ be indexed with numbers $1, ..., L$.

Consider then $F(B,v) = (F_1(B,v), ..., F_N(B,v))$ where the function $F_i(B,v)$ is given by

$$F_i(B,\nu) = \frac{B_i}{1-B_i} - \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i \in \mathcal{N}.$$

with $\nu_{ij} \geq 0$, $(i,j) \in \mathcal{L}$.

We then have

$$\frac{\partial F_i}{\partial B_j} = \frac{1}{(1-B_i)^2} I_{j=i} + \nu_{ij}, \qquad i,j \in \mathcal{N},$$

and

$$\frac{\partial F_i}{\partial \nu_{ij}} = -(1-B_j), \qquad i \in \mathcal{L}, j \in \mathcal{N}_i.$$

Let b be the $N \times N$ diagonal matrix with

$$b_{i,i} = (1-B_i).$$

Furthermore, let R be the $N \times N$ matrix given by

$$R_{i,j} = \begin{cases} \nu_{ij}, & j \in \mathcal{N}_i, \\ 0, & \text{otherwise,} \end{cases}$$

and let T be the $N \times |\mathcal{L}|$ matrix given by

$$T_{i,l} = \begin{cases} (1-B_i)(1-B_j), & l = (i,j), j \in \mathcal{N}_i, \\ 0, & \text{otherwise.} \end{cases}$$

Using the above definitions, we then have that

$$\frac{\partial F}{\partial B} = b^{-2} + R$$

and

$$\frac{\partial F}{\partial v} = -b^{-1}T.$$

Finally, let let

$$\Lambda = (I + bRb)^{-1}.$$

Then we have the following result.

Lemma 7. *The matrix Λ is well-defined and*

$$\sum_{j\in\mathcal{N}} |\Lambda_{ij}| \leq \frac{1}{1-B_*}, \qquad i \in \mathcal{N},$$

where $B_* = \max_{i\in\mathcal{N}} B_i$.

Proof. Recall that

$$\frac{\partial F}{\partial B} = b^{-2} + R$$

which we can rewrite as

$$\frac{\partial F}{\partial B} = b^{-1}(I + bRb)b^{-1}.$$

By Lemma 5, the matrix $\frac{\partial F}{\partial B}$ is invertible. It follows that $(I + bRb)$ is invertible and Λ is well defined.

To show that

$$\sum_{j \in \mathcal{N}} |\Lambda_{ij}| \leq \frac{1}{1 - B_*}, \qquad i \in \mathcal{N},$$

we can use the same argument as given to prove Lemma 1 in [9]. That is, let $M = bRb$, so the diagonal elements of $M_{i,i}$ are all equal to zero and the off-diagonal elements are given by

$$M_{i,j} = (1 - B_i)(1 - B_j)\nu_{ij}.$$

Note that the elements of M are all non-negative and that

$$\sum_{j \in \mathcal{N}} M_{i,j} = (1 - B_i) \sum_{j \in \mathcal{N}_i} \nu_{ij}(1 - B_j) = B_i.$$

Let e denote the vector with all elements being equal to 1. Then we have that

$$Me \leq B_*,$$

where the inequality is understood to be coordinate-by-coordinate. By induction, we obtain for $n \geq 0$ that

$$M^n e \leq B_*^n,$$

and Λ is given by the absolute convergent series

$$\Lambda = \sum_{n=0}^{\infty} (-1)^n M^n.$$

Moreover, for $|\Lambda|$ given by

$$|\Lambda|_{i,j} = |\Lambda_{i,j}|$$

we have

$$|\Lambda| e \leq \sum_{n=0}^{\infty} M^n e \leq \sum_{n=0}^{\infty} B_*^n e = \frac{1}{1 - B_*} e,$$

and the lemma follows. □

Using the above result, we have that

$$\frac{\partial B}{\partial \nu} = (b^{-2} + R)^{-1} b^{-1} T = b \Lambda T.$$

Recall that the solution $(B_i)_{i \in \mathcal{N}}$ to the CSMA fixed point equation is also the solution of the fixed point equation given by

$$\frac{B_i}{1 - B_i} = \sum_{j \in \mathcal{N}_i} \hat{\nu}_{ij}(1 - B_j), \qquad i \in \mathcal{N},$$

where for the constant χ as given in Assumption 1 and for $\kappa = (e - 1)^{-1}\chi$ we have

$$\frac{1}{1 + \kappa\beta} \leq \frac{\hat{\nu}_{ij}}{\nu_{ij}} \leq 1 + \kappa\beta$$

with

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}, \qquad (i,j) \in \mathcal{L}.$$

We use this fact as follows. Let $B(s)$ be the solution to the fixed point equation

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \hat{\nu}_{ij}(s)(1-B_j), \qquad i \in \mathcal{N}, \tag{24}$$

with

$$\hat{\nu}_{ij}(s) = \nu_{ij}(1+\delta_{ij}s), \qquad -1/(1+\kappa\beta) \leq \delta_{ij} \leq 1.$$

Note that as we vary δ_{ij} in the interval $[-1/(1+\kappa\beta), 1]$ and s in the interval $[0, \kappa\beta]$, $\hat{\nu}_{ij}$ will vary in the interval $[1/(1-\kappa\beta), 1+\kappa\beta]$.

Using the chain rule and the fact that

$$\frac{\partial B}{\partial \nu} = b\Lambda T,$$

we obtain for $B_k(s)$, $k = 1, .., N$, that

$$\begin{aligned}
\left|\frac{dB_k}{ds}\right| &= \left|\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i} \frac{dB_k}{d\hat{\nu}_{ij}}\nu_{ij}\delta_{ij}\right| \\
&\leq \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i} \left|\frac{dB_k}{d\hat{\nu}_{ij}}\nu_{ij}\delta_{ij}\right| \\
&\leq (1-B_k)\sum_{i\in\mathcal{N}} |\Lambda_{ki}| \sum_{j\in\mathcal{N}_i} T_{i,(i,j)}|\nu_{ij}\delta_{ji}| \\
&= (1-B_k)\sum_{i\in\mathcal{N}} |\Lambda_{ki}| \sum_{j\in\mathcal{N}_i} (1-B_i)(1-B_j)\nu_{ij}|\delta_{ij}|
\end{aligned}$$

As we have that $\nu_{ij} = 0$ for $j \notin \mathcal{N}_i$, we obtain that

$$\begin{aligned}
\left|\frac{dB_k}{ds}\right| &\leq (1-B_k)\sum_{i\in\mathcal{N}} |\Lambda_{ki}| \sum_{j\in\mathcal{N}_i} (1-B_i)(1-B_j)\nu_{ij}|\delta_{ij}| \\
&= (1-B_k)\sum_{i\in\mathcal{N}} |\Lambda_{ki}|(1-B_i) \sum_{j\in\mathcal{N}} \hat{\nu}_{ij}\left|\frac{\delta_{ij}}{1+s\delta_{ij}}\right|(1-B_j).
\end{aligned}$$

We then have the following result.

Proposition 6. *Let $\kappa = (e-1)^{-1}\chi$ and let $B(s)$ be the solution to the fixed point equation*

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \hat{\nu}_{ij}(s)(1-B_j), \qquad i \in \mathcal{N},$$

with

$$\hat{\nu}_{ij}(s) = \nu_{ij}(1+\delta_{ij}s), \qquad -1/(1+\kappa\beta) \leq \delta_{ij} \leq 1.$$

Then for $0 \leq s \leq \kappa\beta$, we have that

$$(1-B_i(0))e^{-\chi(s+s^2/2)} \leq 1 - B_i(s) \leq (1-B_i(0))e^{\chi(s+s^2/2)}, \qquad i \in \mathcal{N}.$$

Proof. For the proof, we use the same analysis as given to prove Theorem 2 and Corollary 2 in [9]. That is, for $s \in [0, \kappa\beta]$ and $\delta_{ij} \in [-1/(1+\kappa\beta), 1]$ we have

$$-1 \leq \frac{\delta_{ij}}{1+s\delta_{ij}} \leq 1.$$

Combining this bound with the fact that $B_j(s)$ is the solution to Eq. (24), we have that

$$\sum_{j\in\mathcal{N}} \hat{\nu}_{ij}\left|\frac{\delta_{ij}}{1+s\delta_{ij}}\right|(1-B_j) \leq \frac{B_i}{1-B_i}.$$

Combining the above result with Lemma 7, it then follows that

$$\left|\frac{dB_k}{ds}\right| \le (1-B_k)\sum_{i\in\mathcal{N}} |\Lambda_{ki}| B_i \le (1-B_k)\frac{B_*}{1-B_*}. \tag{25}$$

Recall that $B_i(s)$ is the solution to

$$\frac{B_i(s)}{1-B_i(s)} = \sum_{j\in\mathcal{N}} \hat{\nu}_{ij}(s)(1-B_j(s)), \qquad s\in[0,\kappa\beta],$$

with

$$\hat{\nu}_{ij}(s) = \nu_{ij}(1+\delta_{ij}s), \qquad -1/(1+\kappa\beta) \le \delta_{ij} \le 1.$$

As

$$\hat{\nu}_{ij}(s) \le \nu_{ij}(1+s), \qquad -1/(1+\kappa\beta) \le \delta_{ij} \le 1$$

and by Assumption 1 we have that

$$\sum_{j\in\mathcal{N}_i} \nu_{ij} \le \chi,$$

it follows that

$$\frac{B_*}{1-B_*} < \chi(1+s).$$

Combining this result with Eq. (25), we obtain that

$$\left|\frac{dB_k}{ds}\right| \le (1-B_k)\chi(1+s), \qquad s\in[0,\kappa\beta],$$

and the proposition follows. □

We have the following corollary.

Corollary 2. *The solution B_i, $i\in\mathcal{N}$, to the CSMA fixed point equation given by Eq.(20) and (21) satisfies*

$$(1-B_i)e^{-\chi(\kappa\beta+(\kappa\beta)^2/2)} \le 1-\hat{B}_i \le (1-B_i)e^{\chi(\kappa\beta+(\kappa\beta)^2/2)},$$

where $\hat{B}_i$ is the solution to the Erlang fixed point equation

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i\in\mathcal{N},$$

with

$$\nu_{ij} = \frac{p_{(i,j)}+p_{(j,i)}}{\beta}.$$

Proof. Recall that if we vary δ_{ij} in the interval $[-1/(1+\kappa\beta),1]$ and s in the interval $[0,\kappa\beta]$, $\hat{\nu}_{ij}$, then

$$\hat{\nu}_{ij}(s) = \nu_{ij}(1+\delta_{ij}s), \qquad -1/(1+\kappa\beta) \le \delta_{ij} \le 1.$$

will vary in the interval $[1/(1-\kappa\beta), 1+\kappa\beta]$.

The corollary then follows immediately from Proposition 6 and from the fact that CSMA fixed point is a solution to the fixed point equation

$$B_i = \sum_{j\in\mathcal{N}_i} \hat{\nu}_{ij}(1-B_i)(1-B_j), \qquad i\in\mathcal{N},$$

where $\hat{\nu}_{ij}\ge 0$ is such that

$$\frac{1}{1+\kappa\beta} \le \frac{\hat{\nu}_{ij}}{\nu_{ij}} \le 1+\kappa\beta, \qquad (i,j)\in\mathcal{L},$$

and $\hat{\nu}_{ij}=0$ if $(i,j)\notin\mathcal{L}$.

□

The above corollary states that the solution to the CSMA fixed point equation given by Eq.(20) and (21) and the solution to the e Erlang fixed point equation

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i \in \mathcal{N},$$

with

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta},$$

become (asymptotically) identical for large N, as by Assumption 1 we have hat β approaches 0 as N increases. We are going to use this result in Section B.9 to prove Proposition 5.

B.6 Existence of Steady-State Probabilities

In this section, we show that the family of CSMA policies $\mathbf{p}$ is contained in the set $\mathcal{P}$ of all policies that have well-define link service rates.

Consider a CSMA policy $\mathbf{p}$ with sensing period β. Furthermore, recall that $\beta_l(l')$ is the amount of time link l requires to detect that link l' has finished transmitting a packet, i.e. $\beta_l(l')$ is the sensing delay of link l for link l' (see also Appendix A).

We then use the following convention. We say that a node i is idle if node i is currently neither sending, nor receiving, a data packet. We say that a link $l=(i,j)$ is idle if both node i and j are idle. Otherwise, we say that node i (link (i,j)) is busy.

For a given directed link $l=(i,j)$, we refer to node i as the source node of link l. We then say that link $l=(i,j)$ is sensed to be idle by its source node, if node i is (a) currently idle and (b) senses node j to be idle. Otherwise, we say that node i senses link l to be busy.

Suppose that at time t_0 node i has sensed link $l=(i,j)$ to be idle for exactly the duration of a sensing period β, i.e. node i first detect that link l is idle at time $t_0-\beta$. Furthermore, suppose that at time t_0 node i starts a packet transmission on link l. Then we say that link l has been been idle in the interval $[t_0-\beta, t_0)$.

If at time t_0, link $l=(i,j)$ just became busy (either because node i started a packet transmission on link l, or because a link $l' \in \mathcal{I}_l$ that interferes with link l started a packet transmission) and that time t_1 is the first time after time t that link l is idle again, then we refer to the interval $[t_0, t_1)$ as a busy period of link l.

Let $y_l(t)$ indicate whether link l is busy ($y_l(t)=1$) or idle ($y_l(t)=0$). In this section we show that then the steady-state probabilities

$$P(y_i=0) = \lim_{k\to\infty} P(y_i(k\beta)=0), \qquad i \in \mathcal{L},$$

and

$$P(y_i=0, y_j=0) = \lim_{k\to\infty} P(y_i(k\beta)=0, y_j(k\beta)=0), \qquad i,j \in \mathcal{L},$$

exist.

Note that the state of the system at time t can be characterized by the vector $(y(t), z(t))$ where

$$y(t) = (y_l(t))_{l\in\mathcal{L}},$$

indicates for each link $l \in \mathcal{L}$ whether l is busy ($y_l(t)=1$) or not ($y_l(t)=0$), and

$$z(t) = (z_l(t))_{l\in\mathcal{L}},$$

indicates the remaining time until node i has the chance to start a packet transmission on link l (if link l is currently idle), or the time until link l becomes idle again (if link l is currently busy).

The existence of the steady-state probabilities $p(y_i=0)$ and $p(y_i=0, y_i=0)$, $i,j \in \mathcal{N}$, can easily be established for the special case where (a) all sensing delays are equal to β, i.e. we have

$$\beta_l(l') = \beta, \qquad l, l' \in \mathcal{L},$$

and (b) the sensing times of all nodes are aligned, i.e. all nodes are initial idle and start sensing links at time $t_0 = 0$. In this case, the system dynamics are given by a finite-state Markov chain $(y(k), z(k)$, $k \geq 0$, such that

$$(y_l(k), z(k)) = (y_l(k\beta), z_l(k\beta)),$$

where $y_l(k) \in \{0, 1\}$ and

$$z_l(k) \in \{\beta, 2\beta, ..., 1, 1 + \beta\}, \qquad l \in \mathcal{L}, k \geq 0.$$

Furthermore, the Markov chain has a single-recurrent class containing the state $(y*, z^*)$ given by

$$y_l^* = 0 \quad \text{and } z_l^* = \beta, \qquad l \in \mathcal{L},$$

and is aperiodic as the recurrent state (y^*, z^*) has a self-transition. It then follows that the above steady-state probabilities exist.

For the general case where not all sensing times are equal to β, or perfectly aligned, we define a renewal process [7] to establish the existence of the above steady-state probabilities.

Without loss of generality we assume for the rest of this section that

(a) for all links $(i, j) \in \mathcal{L}$ we have that $p_{(i,j)} > 0$, and

(b) the interference graph consists of one connected component, where the vertex set of the interference graph is equal to $\mathcal{L}$ and there exists an edge between two vertices l, l' in the interference graph if link l and l' interfere with each other.

B.6.1 Recurrent State (y^*, z^*)

In the following, we construct a recurrent state (y^*, z^*) that we use to define a renewal process for the general case where not all sensing times are equal to β, or perfectly aligned, . To do this, we first iteratively number the links in the following way. At step 1, let l_1 be an arbitrary link in $\mathcal{L}$ and let S_1 be the set of links that have an interference constraint with link l_1, i.e. we have

$$S_1 = \mathcal{I}_{l_1}.$$

In addition set $B_1 = \{l_1\}$, set $A_1 = S_1$, and set $C_1 = \mathcal{L} \backslash (S_1 \cup \{l_1\})$, i.e. set C_1 contains all links except for link l_1 and the links that interfere with l_1. We then apply this procedure recursively as follows. Suppose that we are given the sets A_k, B_k, and C_k, of step k. Then we proceed as follows at step $k+1$. If the set A_k is empty, then we stop. Otherwise, we pick an arbitrary link from the set A_k and label it as l_{k+1}. Let S_{k+1} be the set of links in set C_k that interfere with link l_{k+1}, i.e. we have

$$S_{k+1} = C_k \cap \mathcal{I}_{l_{k+1}}.$$

Set $B_{k+1} = B_k \cup \{l_{k+1}\}$, set $A_{k+1} = (A_k \cup S_{k+1}) \backslash \{l_{k+1}\}$, and set $C_{k+1} = C_k \backslash S_{k+1}$.

Without loss of generality, we assumed that the interference graph is connected, and the above procedure will terminate after L steps with $A_L = C_L = \emptyset$.

Having labeled the links as given above, we then construct the following sample path of the system to which we will refer to as sample path SP^*.

Sample Path SP^*: *Suppose that during in the interval $[t_0, t_0 + \beta)$ are idle. Then let time*

$$t_0' = t_0 + \beta,$$

and let link l_1 starts a packet transmission at time $t_0' + z_{l_1}(t_0')$ and all other links to remain idle during in the interval $[t_0', t_0' + 2\beta)$). In this case, the packet transmission of link l_1 will not experience a collision. Let $t_1 = t_0' + z_{l_1}(t_0') + 1$ be the time when l_1 finishes its transmission and let all other links remain idle during the interval $[t_0' + 2\beta, t_1)$.

Then proceed iteratively as follows. Let t_k, $k = 1, ..., N$, be the time when link l_k finishes its packet transmission, and let all links to be idle in the interval $[t_k, t_k + \beta)$. Then set

$$t_k' = t_k + \beta,$$

and let link l_{k+1} start a packet transmission at time $t'_k + z_{l_k}(t'_k)$ and all other links to remain idle during in the interval $[t'_k, t'_k + 2\beta)$. Let $t_{k+1} = t'_k + z_{l_k}(t'_k) + 1$ be the time when link l_{k+1} finishes its transmission and let all other links remain idle during the interval $[t'_k + 2\beta, t_{k+1})$.

Let time t_L be the time when link l_L finished its packet transmission and let all links to remain idle in the interval $[t_L, t_L + \beta)$. Finally, let

$$t_r = t_L + \beta + z_{l_1}(t_L)$$

be the time when link l_1 has a chance to start a packet transmission in the interval $[t_L + \beta, t_L + 2\beta)$, given that the source node of link l_1 continues to sense link l_1 to be idle during the interval $[t_r - \beta, t_r)$.

Having defined the sample path SP^*, we show next that the state variable $z(t_r) = (z_l(t_r))_{l\iota\mathcal{L}}$ at the end of the sample path SP^* does not depend on the state $z(t'_0)$ at time t'_0, but is uniquely determined by the sequence of how all links make their transmission attempts and the fact that all links were idle at time t'_0. To do this, let

$$\hat{z}_l(t) = \mod{}_\beta \Big[z_{l_1}(t) - z_l(t) \Big].$$

be the difference (offset) between the time when the current active period ends for link l_1 and l. We have the following result.

Lemma 8. *Let the time t'_k, $k = 1, ..., L$ be as given in the definition of the sample path SP^*. Then at time t'_k, $k = 1, ..., L$, for all links l in the set $A_k \cup B_k$ the offset $\hat{z}_l(t'_k)$ is given by a function that does not depend on $z(t'_0)$, but depends only on the constants $\beta_l(l')$, $l, l' \in \mathcal{L}$, and the sequence of the first k links that are activated in the sample path SP^*.*

Proof. As we do not require the transmission time 1 to be divisible by β, let Δt be given by

$$\Delta t = \mod{}_\beta(1).$$

We prove the lemma by induction. For the sample path SP^*, note that at time t'_1 we have for all links l in the set $A_1 \cup B_1$ that

$$\hat{z}_l(t'_1) = \beta_l(l_1)$$

where $\beta_l(l_1)$ is the time link l requires to sense that link l_1 has finished a packet transmission. Hence the conditions given in the lemma are true for $k = 1$.

Suppose that the lemma is correct for $k - 1 \geq 1$, and let l_k be the link kth link that is activated in the sample path SP^*. We consider the following two cases. First suppose that $l_1 \notin \mathcal{I}_{l_k}$. Then for all links $l \in A_k \cup B_k$ such that $l \notin \mathcal{I}_{l_k}$, we have that

$$\hat{z}_l(t'_k) = \hat{z}_l(t'_{k-1}).$$

For link l_k we have that

$$\hat{z}_{l_k}(t'_k) = \mod{}_\beta \Big[\hat{z}_{l_k}(t'_{k-1}) + \Delta t \Big].$$

Finally, for all links $l \in A_k \cup B_k$ such that $l \in \mathcal{I}_{l_k}$, we have that

$$\hat{z}_l(t'_k) = \mod{}_\beta \Big[\hat{z}_{l_k}(t'_{k-1}) + \Delta t + \beta_l(l_k) \Big].$$

Next suppose that $l_1 \in \mathcal{I}_{l_k}$. Then for link l_k we have that

$$\hat{z}_{l_k}(t'_k) = \beta - \beta_{l_1}(l_k).$$

For all links $l \in A_k \cup B_k$ such that $l \notin \mathcal{I}_{l_k}$, we have that

$$\hat{z}_l(t'_k) = \mod{}_\beta \Big[\beta - \beta_{l_1}(l_k) + \hat{z}_{l_k}(t'_{k-1}) - \hat{z}_l(t'_{k-1}) + \Delta t \Big].$$

Finally, for all links $l \in A_k \cup B_k$ such that $l \in \mathcal{I}_{l_k}$, we have that

$$\hat{z}_l(t'_k) = \mod{}_\beta \Big[\beta_{l_1}(l_k) - \beta_l(l_k) \Big].$$

As by the induction hypothesis $\hat{z}_l(t'_k)$ does not depend on $z(t'_0)$ but only on the constants $\beta_l(l')$, $l, l' \in \mathcal{L}$, and the sequence of the first k links that activated in the sample path SP^*, the statement of the lemma is true for step k. The results then follows. □

We then have the following lemma.

Lemma 9. *Let t'_0 and t_r be as given in the definition of the sample path SP^*. The state $(y^*, z^*) = (y(t_r), z(t_r))$ in the sample path SP^* is given by a function that does not depend on $(y(t'_0), z(t'_0))$, but only on the constants $\beta_l(l')$, $l, l' \in \mathcal{L}$, and the sequence of links activated in the sample path SP^**

Proof. This result follows immediately from Lemma 8 and the fact that

$$z_{l_1}(t_r) = 0$$

and

$$z_l(t_r) = \hat{z}_l(t'_L), \qquad l \neq l_1.$$

$\square$

Next we show that there exists a positive constant p_0 such that the probability that the above sample path reaches state (y^*, z^*) within at most $(1 + L)(1 + 2\beta)$ time units is lower-bounded by p_0.

Lemma 10. *Let*

$$p_{\max} = \max_{(i,j)\in\mathcal{L}} p_{(i,j)}$$

and

$$p_{\min} = \min_{(i,j)\in\mathcal{L}} p_{(i,j)}.$$

Then the probability that we reach the state (y^*, z^*) within $(1 + L)(1 + 2\beta)$ time units from any given initial state $(y(t_0), z(t_0)$ is lower-bounded by

$$p_0 = (1 - p_{\max})^{L(\lceil 1/\beta\rceil+2)} \Big[p_{\min}(1 - p_{\max})^{L(2+\lceil 1/\beta\rceil}\Big]^L.$$

Proof. Note that from any initial state $(y(t_0), z(t_0)$, with probability at least

$$(1 - p_{\max})^{L(\lceil 1/\beta\rceil+2)}$$

we have for

$$t'_0 = t_0 + 1 + 2\beta$$

that all links are idle during the interval $[t'_0 - \beta, t'_0)$.

Consider the sample path SP^*. The probability that link l_1 starts a packet transmission in the interval $[t'_0, t'_0 + \beta)$ and all other links remain idle in the interval $[t'_0, t'_0 + 2\beta)$, is lower-bounded by

$$p_{\min}(1 - p_{\max})^{2L}.$$

The probability that no other link starts a packet transmission in the interval $[t'_0 + 2\beta, t'_0 + 1 + 2\beta)$ is lower-bounded by

$$(1 - p_{\max})^{L\lceil 1/\beta\rceil}$$

Let t_1 be the time when link l_1 finishes its packet transmission; note that

$$t_1 < t'_0 + \beta + 1.$$

If all other links remain idle during the interval $[t'_0, t'_0 + 1 + 2\beta)$, then all links are idle during the interval $[t_1, t_1 + \beta)$.

The result follows by applying the above argument iteratively to the case where link l_k, $k = 2, ..., L$, start a packet transmission under the sample path SP^*. $\square$

B.6.2 Renewal Process

Using Lemma 10, we can define a renewal process where renewal epochs are marked by visits to the recurrent state (y^*, z^*).

Lemma 11. *The expected length of the interval between visits to state (y^*, z^*) is bounded, and the visits to the state (y^*, z^*) define a renewal process.*

We have the following result for the resulting renewal process.

Lemma 12. *The renewal process defined by visits to the state (y^*, z^*) is either aperiodic, or has a period β/c where c is a positive integer.*

Proof. The lemma follows immediately from the fact that if $(y(t_0), z(t_0)) = (y^*, z^*)$ then with probability at least $(1 - p_{max})^L$ we have that

$$(y(t_0 + \beta), z(t_0 + \beta)) = (y^*, z^*).$$

□

Combining the above lemmas, we obtain the following result.

Proposition 7. *For every sensing period $\beta > 0$, the family of CSMA policies $\mathbf{p}$ is contained in the set $\mathcal{P}$ of all policies that have well-define link service rates.*

Proof. Let $I_{(i,j)}(t)$ be the in indicator function for whether link (i, j) is transmitting at time t a packet that does not experience a collision during its entire transmission time. Using Lemma 12, we then we have that (see for example [7])

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t I_{(i,j)}(\tau) d\tau = \lim_{k\to\infty} P(I_{(i,j)}(k\beta) = 1).$$

□

B.7 Properties of Balance Equations

In this section, we characterize the balance equations for the steady-state probabilities

$$P(y_i = 0) = \lim_{k\to\infty} P(y_i(k\beta) = 0), \qquad i \in \mathcal{L},$$

and

$$P(y_i = 0, y_j = 0) = \lim_{k\to\infty} P(y_i(k\beta) = 0, y_j(t) = 0), \qquad i, j \in \mathcal{L},$$

under a CSMA policy $\mathbf{p}$ with sensing period β.

We are going to use the following notation. If node i is busy at time t, i.e. if $y(t) = 1$, let $x_i(t)$, $i \in \mathcal{N}$, denote the time until node i becomes idle again, i.e. until i stops sending, or receiving, the current packet transmission. Furthermore, if node i and j are jointly idle at time t, i.e. we have that $y_i(t) = y_j(t) = 0$, then let $x_{ij}(t) = x_{ji)}(t)$ be the amount of time that node i and j haven been jointly idle. Note that if node i and j have to be jointly idle for at least the duration of sensing period β before node i can potentially start a packet transmission on link (i, j).

B.7.1 Preliminary Lemmas

For a given link $l = (i, j)$, recall that $\mathcal{I}_l$ be the set of links that interfere with l. Suppose that at time t node i and j have been jointly idle for at least β time units, i.e. we have that $y_i(t) = y_j(t) = 0$ and $x_{ij}(t) \geq \beta$. Given a CSMA policy $\mathbf{p}$, the probability that node i starts a packet transmission on link l during the interval $(t, t + \beta]$ is then lower-bounded by

$$p_{(i,j)} \prod_{l' \in N_l} (1 - p_{l'}),$$

upper bounded by $p_{(i,j)}$.

Note that from the definition of a CSMA policy, it immediately follows that $p_{(i,j)}$ is an upper-bound on the probability that node i starts a packet transmission on link l during the interval $(t, t + \beta]$. To see that $p_{(i,j)} \prod_{l' \in N_l} (1 - p_{l'})$ is lower-bound, we observe the following. Given that at time t node i and j have been jointly idle for at least β time units, let t_0 be the earliest time after t when node i has the chance to start a packet transmission on link l, if link l remains idle in the interval (t, t_0). Note that

$$t_0 \leq t + \beta.$$

In the worst case, all links $l' \in \mathcal{I}_l$ have an opportunities to start a packet transmission in the interval $[t_0 - \beta, t_0)$. In this case, the probability that that no link $l' \in \mathcal{I}_l$ starts a packet transmission during the interval $[t_0 - \beta, t_0)$, and link l has the opportunity to start a packet transmission at time t_0 is lower-bounded by

$$\prod_{l' \in \mathcal{I}_l} (1 - p_{l'})$$

and the probability that that link l starts a packet transmission in the interval $(t, t+\beta]$ is lower-bounded by $p_{(i,j)} \prod_{l' \in \mathcal{I}_l} (1 - p_{l'})$.

We have the following result.

Lemma 13. *Suppose that at time t node i and j have been jointly idle for at least β time units, i.e. we have that $y_i(t) = y_j(t) = 0$ and $x_{ij}(t) \geq \beta$. Then there exists a constant κ_p such that the probability that the link starts a packet transmission in the interval $(t, t + \beta]$ is lower-bounded by*

$$\frac{1}{1 + \kappa_p \beta} p_{(i,j)}, \qquad \beta \in [0, (4\chi)^{-1}]$$

and upper-bounded by

$$(1 + \kappa_p \beta) p_{(i,j)}.$$

Proof. For $k \in \mathcal{I}_l$ we have that

$$\left| \frac{d}{dp_k} p_{(i,j)} \prod_{l' \in \mathcal{I}_l} (1 - p_{l'}) \right| \leq p_{(i,j)}.$$

From the mean value theorem, in then follows that

$$p_{(i,j)} (1 - \sum_{l' \in \mathcal{I}_l} p_{l'}) \leq p_{(i,j)} \prod_{l' \in \mathcal{I}_l} (1 - p_{l'}).$$

By Assumption 1 we have that

$$\sum_{l' \in \mathcal{I}_l} p_{l'} \leq 2\chi\beta,$$

and it follows that

$$p_{(i,j)} (1 - 2\chi\beta) \leq p_{(i,j)} \prod_{l' \in \mathcal{I}_l} (1 - p_{l'}).$$

Note that for

$$\kappa_p \geq 4\chi$$

we have that

$$\frac{1}{1 + \kappa_p \beta} \leq (1 - 2\chi\beta), \qquad \beta \in [0, (4\chi)^{-1}].$$

The result then follows. ∎

Below, we derive additional lemmas that we are going to use in Section B.7.2.

Lemma 14. *The probability that a packet transmission experiences a collision is upper-bounded by 4χ.*

Proof. Suppose that node i starts a packet transmission on link $l = (i, j)$ at time t. Then this packet transmission will experience a collision only if another node starts a packet transmission on a link $l' \in \mathcal{I}_l$ in the interval $(t - \beta, t + \beta)$. This is because by Assumption 2, we have that for links $l' \in \mathcal{I}_l$ we have that the sensing delay $\beta_l(l')$ and $\beta_{l'}(l)$ is bounded by β. Furthermore, by Assumption 1 we have that

$$\sum_{l' \in \mathcal{I}_l} p_{l'} \leq 2\chi,$$

and the lemma follows. □

Lemma 15. *We have*

$$P(y_i = 1, x_i \in (0, \beta]) = P(y_i = 1, x_i \in (1 - \beta), 1]).$$

Proof. The above lemma follows immediately from the fact that a packet transmission takes 1 time unit. □

Lemma 16. *We have*

$$P(y_i = 1, x_i \in (0, \beta])\frac{1}{\beta} \leq P(y_i = 1) \leq P(y_i = 1, x_i \in (0, \beta])\frac{1 + 2\beta}{\beta}.$$

Proof. The results follows immediately from the fact that the length of a busy period is bounded between 1 (the length of a successful transmission) and $1 + 2\beta$ (the maximal length of a collision). □

Lemma 17. *We have*

$$P(y_i = 1, x_i \in (1, 1 + \beta]) \leq P(y_i = 1, x_i \in (1 - \beta, 1])4\chi.$$

Proof. Note that the event $\{y_i = 1, x_i \in (1, 1 + \beta]\}$ indicates that a packet transmission resulted in a collision. By Lemma 14, the probability of this happening is upper-bounded by 4χ, and the lemma follows. □

Lemma 18. *We have*

$$\frac{P(y_i = 0, y_j = 0, x_{ij} \geq \beta)}{P(y_i = 0, y_j = 0)} \geq (1 - 4\chi\beta), \qquad i, j \in \mathcal{N}.$$

Proof. Suppose that that at time t node i and j have just become jointly idle, and let T_t the time it takes starting from t until either node i or j become busy. Note that by Assumption 1, we have that

$$E[T_t] \geq \beta\frac{1}{2\chi\beta} - \beta.$$

It then follows that

$$\frac{P(y_i = 0, y_j = 0, x_{ij} \geq \beta)}{P(y_i = 0, y_j = 0)} \geq \frac{\beta/2\chi\beta - 2\beta}{\beta/2\chi\beta - \beta} = \frac{1 - 4\chi\beta}{1 - 2\chi\beta} \geq 1 - 4\chi\beta.$$

□

B.7.2 Bounds on the Steady-State Probabilities

In the following, we derive bounds on the steady-state probability $P(y_i = 1)$, $i \in \mathcal{N}$. We start with the following lemma.

Lemma 19. *For*

$$\beta \in [0, (16\chi)^{-1}]$$

there exists a constant κ'_p *such that*

$$\begin{aligned}
&\frac{1}{1 + \kappa'_p\beta} \sum_{j \in \mathcal{N}_i} P(y_i = 0, y_j = 0)(p_{(i,j)} + p_{(j,i)}) \\
&\leq P(y_i = 1, x_i \in (1 - \beta, 1]) \leq \\
&(1 + \kappa'_p\beta) \sum_{j \in \mathcal{N}_i} P(y_i = 0, y_j = 0)(p_{(i,j)} + p_{(j,i)}).
\end{aligned}$$

Proof. Suppose that the system is in steady-state at time t_0 and that we observe the evolution of the system from time t_0 to $t_0+\beta$. Using lemma 12 which states that the renewal process is either aperiodic, or has a period of β/c where c is a positive integer, it follows that at time $t_0+\beta$ the system is again in steady-state. Furthermore, suppose that at time t_0 nodes i and j have been jointly idle for at least β time units, i.e. we have that $y_i(t_0)=y_j(t_0)=0$ and $x_{ij}(t_0)\geq\beta$. Then by Lemma 13, for $\beta\in[0,((4\chi\beta)^{-1})$ there exists a constant κ_p such that the probability that link (i,j) starts a packet transmission during the interval $(t_0,t_0+\beta]$ is bounded between $\frac{1}{(1+\kappa_p\beta)}p_{(i,j)}$ and $(1+\kappa_p\beta)p_{(i,j)}$. Furthermore, by Lemma 14 the probability that this transmission will result in a collision is upper-bounded by $4\chi\beta$. When the transmission does not result in a collision, then at $t_0+\beta$ the remaining time until node i finishes the packet transmission will be in the interval $(1-\beta,1]$, i.e. we have $x_i(t_0+\beta)\in(1-\beta,1]$.

Combining the above results, we obtain the following inequality

$$\frac{1-4\chi\beta}{1+\kappa_p\beta}\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0,x_{ij}\geq\beta)(p_{(i,j)}+p_{(j,i)})\leq P(y_i=1,x_i\in(1-\beta,1])$$

and

$$\begin{aligned}P(y_i=1,x_i\in(1-\beta,1]) &\leq 1+\kappa_p\beta)\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)(p_{(i,j)}+p_{(j,i)})+...\\ &\quad +P(y_i=1,x_i\in(1,1+\beta]).\end{aligned}$$

Using Lemma 18, we obtain for the first inequality that

$$\frac{(1-4\chi\beta)^2}{1+\kappa_p\beta}\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)(p_{(i,j)}+p_{(j,i)})\leq P(y_i=1,x_i\in(1-\beta,1])$$

Furthermore, using Lemma 17 we obtain that

$$\begin{aligned}P(y_i=1,x_i\in(1-\beta,1]) &\leq 1+\kappa_p\beta)\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)(p_{(i,j)}+p_{(j,i)})+...\\ &\quad +P(y_i=1,x_i\in(1-\beta,1])4\chi\beta,\end{aligned}$$

or

$$P(y_i=1,x_i\in(1-\beta,1])\leq\frac{1+\kappa_p\beta}{1-4\chi}\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)(p_{(i,j)}+p_{(j,i)}).$$

Note that for $\beta\in[0,(16\chi)^{-1}]$ and $\kappa_p'\geq 2(\kappa_p+8\chi)$ we have that

$$\frac{1}{1+\kappa_p'\beta}\leq\frac{1-8\chi\beta}{1+\kappa_p\beta}\leq\frac{(1-4\chi\beta)^2}{1+\kappa_p\beta}.$$

The lemma then follows. $\square$

Using Lemma 19, we obtain the following bound for the steady-state probability $P(y_i=1)$, $i\in\mathcal{N}$.

Lemma 20. *For*

$$\beta\in[0,(16\chi)^{-1}]$$

there exists a constant κ_s such that

$$\frac{1}{1+\kappa_s\beta}\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)\nu_{ij}\leq P(y_i=1)\leq(1+\kappa_s\beta)\sum_{j\in\mathcal{N}_i}P(y_i=0,y_j=0)\nu_{ij},$$

where

$$\nu_{ij}=\frac{p_{(i,j)}+p_{(j,i)}}{\beta}.$$

Proof. Using Lemma 15-19, for $\beta \in [0, (16\chi)^{-1}]$ we have

$$P(y_i = 1, x_i \in (1-\beta, 1])\frac{1}{\beta} \le P(y_i = 1) \le P(y_i = 1, x_i \in (1-\beta, 1])\frac{1+2\beta}{\beta},$$

and there exists a constant κ'_p such that

$$\begin{aligned}
&\frac{1}{1+\kappa'_p\beta}\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)(p_{(i,j)} + p_{(j,i)}) \\
&\le P(y_i = 1, x_i \in (1-\beta, 1]) \le \\
&(1+\kappa'_p\beta)\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)(p_{(i,j)} + p_{(j,i)}).
\end{aligned}$$

Combing the above results, we have that

$$\begin{aligned}
&\frac{1}{1+\kappa'_p\beta}\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)\nu_{ij} \\
&\quad\le P(y_i = 1) \\
&\quad\le (1+\kappa'_p\beta)(1+2\beta)\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)\nu_{ij},
\end{aligned}$$

where

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}.$$

Note that for $\beta \in [0, (16\chi)^{-1}]$ and

$$\kappa_s \ge \kappa'_p + 2 + \frac{\kappa'_p}{8\chi}$$

we have that

$$(1+\kappa'_p\beta)(1+2\beta) \le 1 + \kappa_s\beta.$$

The lemma then follows. ∎

B.8 Characterization of the steady-state probabilities

In this section, we characterize the steady-state probabilities

$$\bar{B}_i = 1 - P(y_i = 0), \qquad i \in \mathcal{N},$$

that a node i is busy under a CSMA policy $\mathbf{p}$ with sensing period β, using the same analysis as given by Hajek and Krishna in Section 3 and 4 of the reference [9] with only minor changes.

Throughout this section, we set

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}, \qquad i, j \in \mathcal{N},$$

with $\nu_{ij} = 0$ if $(i,j) \notin \mathcal{L}$ and $(j,i) \notin \mathcal{L}$.

Note that by Lemma 20 there exists a constant κ_s such that

$$\begin{aligned}
&\frac{1}{1+\kappa_s\beta}\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)\nu_{ij} \\
&\quad\le P(y_i = 1) \\
&\quad\le (1+\kappa_s\beta)\sum_{j\in\mathcal{N}_i} P(y_i = 0, y_j = 0)\nu_{ij}.
\end{aligned}$$

We have the the following result.

Lemma 21. *Let κ_s be the constant of Lemma 20. Then we have that*

$$\begin{aligned}&\frac{1}{1+\kappa_s\beta}\sum_{j\neq k,l} P(y_k=0,y_j=0,y_l=0)\nu_{kj} \leq \\ &P(y_k=1,y_l=0) \leq \\ &(1+\kappa_s\beta)\sum_{j\neq k,l} P(y_k=0,y_j=0,y_l=0)\nu_{kj}.\end{aligned}$$

Proof. Note that we have

$$P(y_k=1,y_l=0)=P(y_k=1|y_l=0)P(y_l=0)$$

and

$$P(y_k=0,y_j=0,y_l=0)=P(y_k=0,y_j=0|y_l=0)P(y_l=0).$$

Therefore, in order to obtain the result if suffices to show that

$$\begin{aligned}&\frac{1}{1+\kappa_s\beta}\sum_{j\neq k,l} P(y_k=0,y_j=0|y_l=0)\nu_{kj} \leq \\ &P(y_k=1|y_l=0) \leq \\ &(1+\kappa_s\beta)\sum_{j\neq k,l} P(y_k=0,y_j=0|y_l=0)\nu_{kj}.\end{aligned}$$

The above inequalities are obtained by the same argument as given in the proof for Lemma 20. $\square$

We then have the following result.

Proposition 8. *Let κ_s be the constant of Lemma 20. We then have that*

$$\frac{1}{1+2r_p}\left(\frac{1}{1+\kappa_s\beta}\right)^{2N} \leq \frac{P(y_i=0,y_j=0)}{P(y_i=0)P(y_j=0)} \leq (1+\kappa_s\beta)^{2N}\,(1+2r_p),$$

where

$$r_p=\frac{p_{max}}{\beta}$$

and p_{max} is as given in Assumption 1.

Proof. Let Z_i be the steady-stated probability $P(y_i=0)$ that node i is idle, let Z_{ij} be the steady-stated probability $P(y_i=0,y_j=0)$ that nodes i and j are jointly idle, and let Z_{ijk} be the steady-stated probability $P(y_i=0,y_j=0,y_k=0)$ that nodes i, j, and k, are jointly idle

We use a proof by induction on the number of nodes in the network, as given in [9]. For a network with $N=1$ node the proposition is trivially true, and suppose that $N\geq 2$.

Using Lemma 21, we have that

$$\frac{1}{1+\kappa_s\beta}\Big(Z_{kl}+\sum_{j\neq k,l} Z_{jkl}\nu_{jk}\Big) \leq Z_l \leq (1+\kappa_s\beta)\Big(Z_{kl}+\sum_{j\neq k,l} Z_{jkl}\nu_{jk}\Big).$$

Furthermore, starting with the equation

$$1=P(y_k=0)+P(y_k=1)$$

and using the result from Lemma 20 which states that

$$\frac{1}{1+\kappa_s\beta}\Big(\sum_{j\in\mathcal{N}_k} P(y_k=0,y_j=0)\nu_{kj}\Big) \leq P(y_k=1) \leq (1+\kappa_s\beta)\Big(\sum_{j\in\mathcal{N}_k} P(y_k=0,y_j=0)\nu_{kj}\Big),$$

we obtain that

$$\begin{aligned}
&\frac{1}{1+\kappa_s\beta}\Big(P(y_k=0)+\sum_{j\in\mathcal{N}_k}P(y_k=0,y_j=0)\nu_{kj}\Big)\\
&\quad\le 1\\
&\quad\le (1+\kappa_s\beta)\Big(P(y_k=0)+\sum_{j\in\mathcal{N}_k}P(y_k=0,y_j=0)\nu_{kj}\Big).
\end{aligned}$$

Combining the above inequalities, we obtain by the same approach as in [9] that

$$\begin{aligned}
&\frac{1}{(1+\kappa_s\beta)^2}\frac{(Z_k+Z_{kl}\nu_{kl})Z_{kl}+\sum_{j\neq k,l}Z_{jk}Z_{kl}\nu_{jk}}{Z_kZ_{kl}+\sum_{j\neq k,l}Z_kZ_{jkl}\nu_{jk}}\\
&\le \frac{P(y_i=0,y_j=0)}{P(y_i=0)P(y_j=0)}\le\\
&(1+\kappa_s\beta)^2\frac{(Z_k+Z_{kl}\nu_{kl})Z_{kl}+\sum_{j\neq k,l}Z_{jk}Z_{kl}\nu_{jk}}{Z_kZ_{kl}+\sum_{j\neq k,l}Z_kZ_{jkl}\nu_{jk}}.
\end{aligned}$$

Using the fact that $Z_{kl}\le Z_k$ and by Assumption 1 we have

$$0\le\nu_{ij}\le 2r_p,$$

it follows that

$$1\le\frac{Z_k+Z_{kl}\nu_{kl}}{Z_k}\le 1+2r_p.$$

Furthermore, by the induction hypotheses applied to the network with $N-1$ nodes that we obtain by deleting node k, we have that

$$\frac{1}{1+2r_p}\left(\frac{1}{1+\kappa_s\beta}\right)^{2(N-1)}\le\frac{Z_{jk}Z_{kl}}{Z_kZ_{jkl}}\le(1+\kappa_s\beta)^{2(N-1)}(1+2r_p).$$

Combining the above results, we obtain that

$$\frac{1}{1+2r_p}\left(\frac{1}{1+\kappa_s\beta}\right)^{2N}\le\frac{P(y_i=0,y_j=0)}{P(y_i=0)P(y_j=0)}\le(1+\kappa_s\beta)^{2N}(1+2r_p),$$

and the result follows. □

We then obtain the following corollary.

Corollary 3. *Let κ_s be the constant of Lemma 20, and let $\bar{B}_i$ be the actual steady-state probability that node i is busy. Then*

$$\frac{\bar{B}_i}{1-\bar{B}_i}=\sum_{j\in\mathcal{N}_i}\tilde{\nu}_{ij}(1-\bar{B}_j)$$

where $\tilde{\nu}_{ij}$ is such that

$$\frac{1}{1+2r_p}\left(\frac{1}{1+\kappa_s\beta}\right)^{2N+1}\le\frac{\tilde{\nu}_{ij}}{\nu_{ij}}\le(1+\kappa_s\beta)^{2N+1}(1+2r_p),$$

where r_p is as given in Proposition 8.

The above results follows immediately from Proposition 8 and Lemma 20. Combining the above corollary with Proposition 6 from Section B.5, we obtain the following result.

Corollary 4. *Let κ_s be the constant of Lemma 20. The actual steady-state probability $\bar{B}_i$, $i\in\mathcal{N}$ that node i is busy satisfies*

$$(1-\hat{B}_i)e^{-\chi(r+r^2/2)}\le 1-\bar{B}_i\le(1-\hat{B}_i)e^{\chi(r+r^2/2)},$$

where $\hat{B}_i$ is the solution to the Erlang fixed point equation given by

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i \in \mathcal{N},$$

where

$$r = (2N+1)(\kappa_s\beta) + 2r_p$$

and r_p is as given in Proposition 8.

B.9 Proof of Proposition 5

In this section, we combine the results of Sections B.5 and B.8 to prove Proposition 5.

Consider a CSMA policy $\mathbf{p}$ for a wireless network consisting of N nodes and set

$$\nu_{ij} = \frac{p_{(i,j)} + p_{(j,i)}}{\beta}, \qquad i,j \in \mathcal{N}.$$

Let B_i, $i = 1,...,N$, be the CSMA fixed point given by Eq. (20) and (21), and let $\bar{B}_i$ be the actual steady-state probability that node i is busy. Then by Corollary 4, we have that steady-state probabilities $\bar{B}_i$, $i \in \mathcal{N}$, satisfy

$$(1-\hat{B}_i)e^{-\chi(r+r^2/2)} \leq 1-\bar{B}_i \leq (1-\hat{B}_i)e^{\chi(r+r^2/2)},$$

where $\hat{B}_i$ is the solution to the Erlang fixed point

$$\frac{B_i}{1-B_i} = \sum_{j\in\mathcal{N}_i} \nu_{ij}(1-B_j), \qquad i \in \mathcal{N},$$

where

$$r = (2N+1)(\kappa_s\beta) + 2r_p$$

and r_p is as given in is as given in Proposition 8.

Furthermore, by Proposition 6 we have that there exists a constant κ such that

$$(1-\hat{B}_i)e^{-\chi(\kappa\beta+(\kappa\beta)^2/2)} \leq 1-B_i \leq (1-\hat{B}_i)e^{\chi(\kappa\beta+(\kappa\beta)^2/2)}.$$

Combining these two results, we immediately obtain Proposition 5.