PERCOLATION IN SELFISH SOCIAL NETWORKS

Charles Bordenave

UC Berkeley

joint work with **David Aldous** (UC Berkeley)

PARADIGM OF SOCIAL NETWORKS

OBJECTIVE

- Understand better the dynamics of social behaviors in large social systems,
- Design computer networks which interact positively with the social behaviors of end users.

POSTULATE

- large number of social agents,
- social relations are based on local decisions,
- the social capital of an agent is related to the graph spanned by its relationships,
- agents are selfish and greedy.

SOCIAL GRAPH AND SOCIAL PAYOFF

V is a set of vertices with pairwise weights $(w_e), e = (u, v) \in V \times V$.

For a graph G on V and a vertex $u \in V$, the payoff of u in G is

$$J_G(u) = \sum_{v \in V} \gamma^{d_G(u,v)} - \sum_{v \sim_G u} w_{(u,v)},$$

with $0 < \gamma < 1$ [Jackson and Wolinsky (1996)].

SOCIAL WELFARE AND THE PRICE OF ANARCHY

The graph maximizing the average payoff of the agents is the social welfare.

But, taking into account:

$\longrightarrow$ the selfishness of the agents

$\longrightarrow$ the local nature of the social graph

this social welfare may be out of reach.

$\Longrightarrow$ With an appropriate notion of stability, only the social graphs in Nash equilibrium are relevant.

The price of anarchy is the difference between the average payoff of the social welfare and the worst case Nash equilibrium [Roughgarden].

STABILITY

A graph G is stable if

(i) for all all edges $e = (u, v)$ in G,

$$J_{G \setminus \{e\}}(u) \leq J_G(u) \quad \text{AND} \quad J_{G \setminus \{e\}}(v) \leq J_G(v)$$

$\longrightarrow$ a vertex may remove an adjacent edge unilaterally,

(ii) for all edges $e = (u, v)$ not in G,

$$J_{G \cup \{e\}}(u) \leq J_G(u) \quad \text{OR} \quad J_{G \cup \{e\}}(v) \leq J_G(v)$$

$\longrightarrow$ a pair of vertices may add an adjacent edge bilaterally.

[Jackson and Wolinsky (1996)], [Johnson and Gilles (2000)].

GREEDY AND SELFISH RECURSION

Start from $G_0 = G_\emptyset$ the fully isolated graph on V. Define the greedy and selfish (GS) recursion as the sequence of graphs $(G_n)_{n\in\mathbb{N}}$ where G_{n+1} is obtained from G_n by the rule:

$$e = (u, v) \in G_{n+1} \backslash G_n \quad \text{if}$$

$$J_{G_n \cup \{e\}}(u) > J_{G_n}(u) \;\; \text{and} \;\; J_{G_n \cup \{e\}}(v) > J_{G_n}(v)$$

$$e = (u, v) \in G_n \backslash G_{n+1} \quad \text{if}$$

$$J_{G_n \backslash \{e\}}(u) > J_{G_n}(u) \;\; \text{or} \;\; J_{G_n \backslash \{e\}}(v) > J_{G_n}(v).$$

SIMPLIFICATION OF THE PAYOFF FUNCTION

For $\gamma = 1$, the payoff reduces to

$$J_G(u) = |C_G(u)| - \sum_{v \sim_G u} w_{(u,v)},$$

where $C_G(u)$ is the connected component of u in G.

PERCOLATION IN THE SELFISH SOCIAL NETWORK

Assume that the vertex set V is infinite. Does it happen that

$$\lim_{n\to\infty} J_{G_n}(u) = +\infty\ ?$$

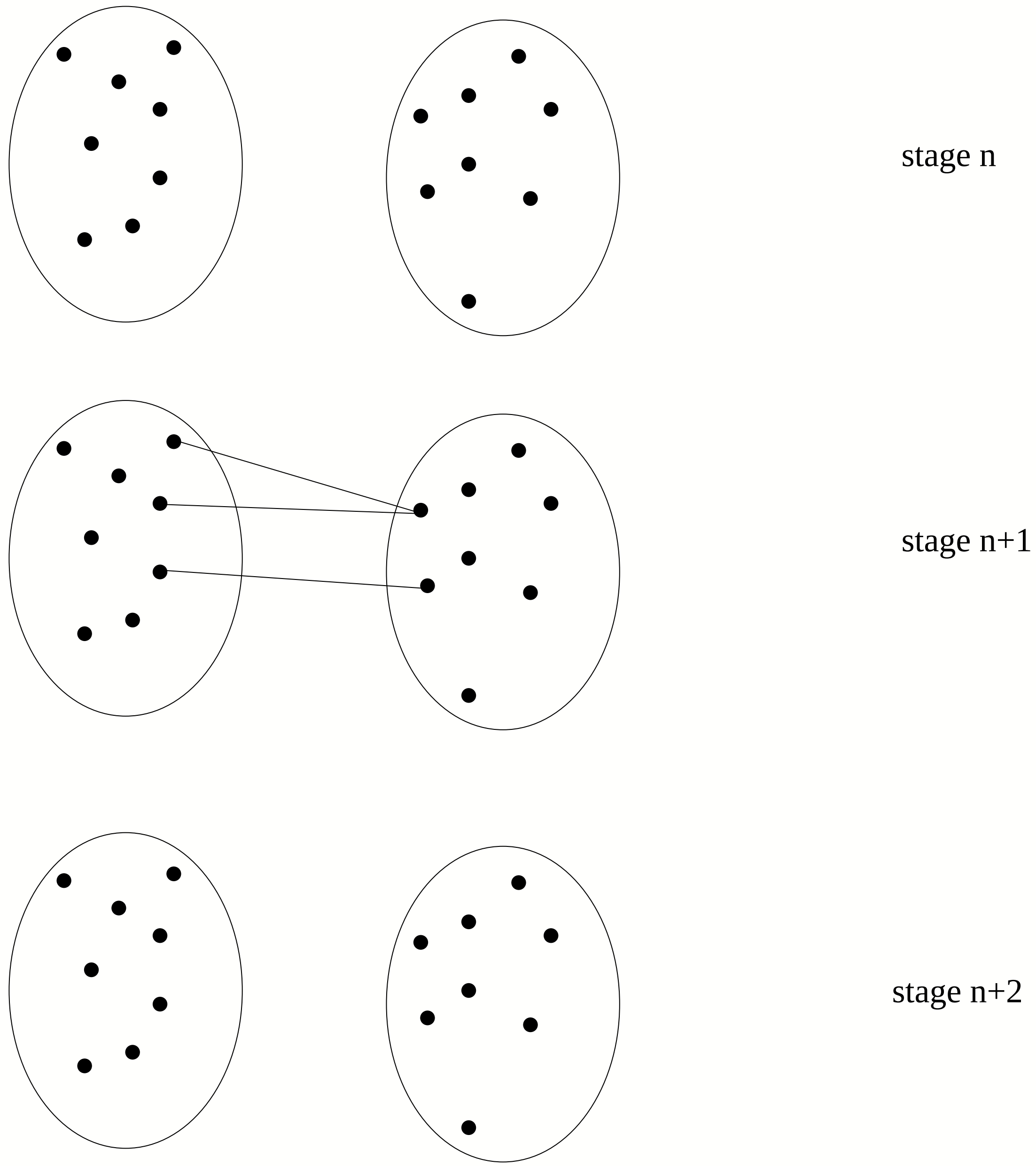
In that case, $\limsup_n C_{G_n}(u)$ is infinite and there is percolation phenomenon in the network.

Percolation $\longleftrightarrow$ low price of anarchy

No percolation $\longleftrightarrow$ high price of anarchy.

If $|V|$ finite but goes to infinity, same kind of statement.

UNWANTED ARTIFACT IN THE MODEL



CONTINUOUS TIME GS RECURSION

(i) The rule of addition is unchanged: vertices add edges at time $n = 0, 1, \cdots$ and

$$e = (u, v) \in G_{n+} \backslash G_n \text{ if } J_{G_n \cup \{e\}}(u) > J_{G_n}(u) \text{ and } J_{G_n \cup \{e\}}(v) > J_{G_n}(v)$$

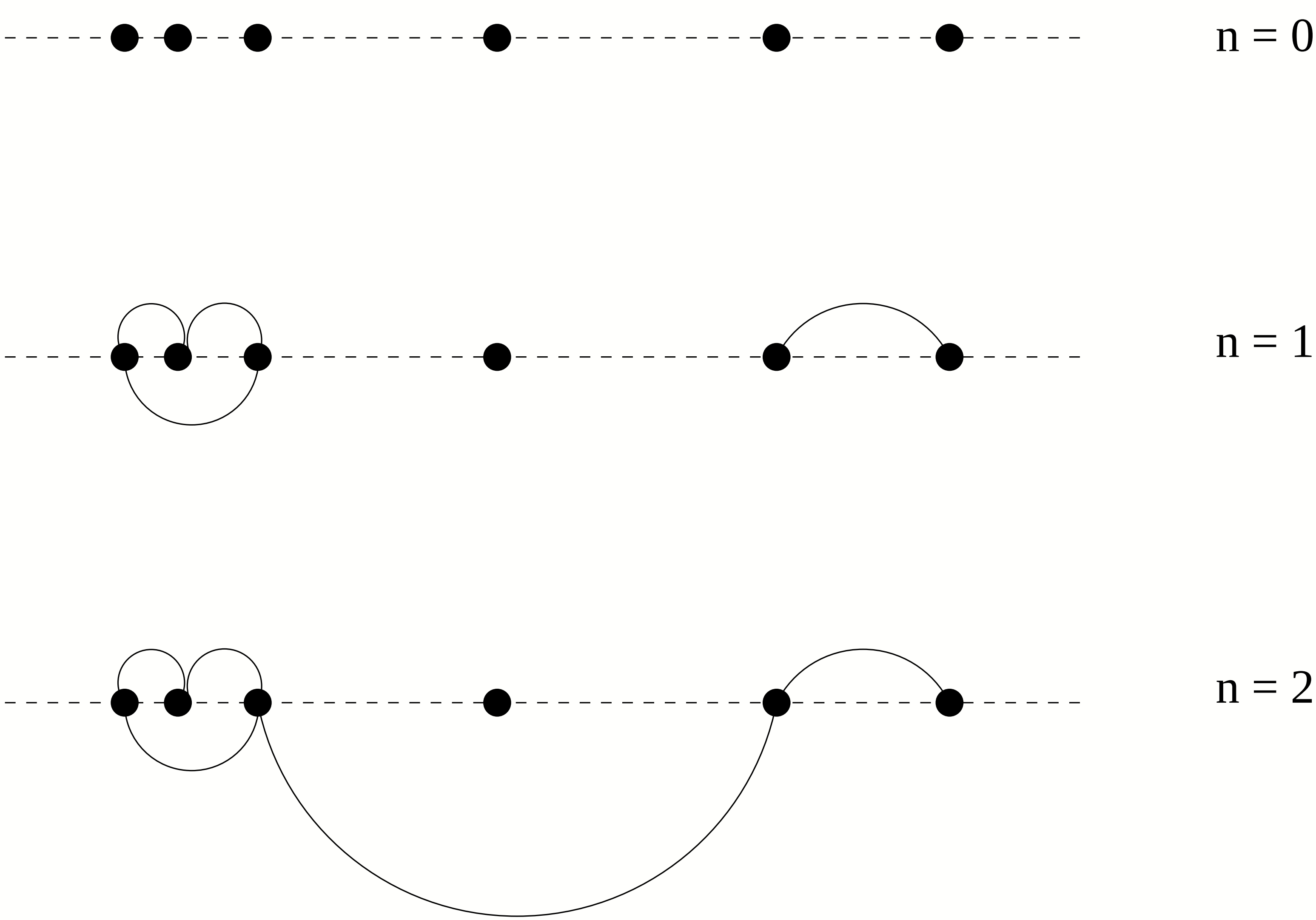
(ii) To each vertex $u \in V$, we associate an independent Poisson process on $\mathbb{R}_+$, $0 = T_0^u < T_1^u < \cdots$. At time T_i^u, the vertex u removes its adjacent edges according to the rule:

$$e = (u, v) \in G_{T_i^u} \backslash G_{T_i^u +} \text{ if } J_{G_{T_i^u} \backslash \{e\}}(u) > J_{G_{T_i^u}}(u).$$

Easy to check that almost surely for all $u \in \mathbb{N}$, the sequence $(C_{G_n}(u))_{n \in \mathbb{N}}$ is a non-decreasing sequence of sets, and $(J_{G_n}(u))_{n \in \mathbb{N}}$ is non-decreasing.

$\Longrightarrow$ In order to analyze $|C_{G_n}(u)|$, only the rule (i) may be considered.

EXAMPLE OF A SPATIAL SOCIAL NETWORK



POISSON MODEL

- The vertex set $V = \{\eta_i\}_{i\in\mathbb{N}}$, where under the probability P_λ, $\Phi = \sum_i \delta_{\eta_i}$ is a Poisson point process of intensity λ on $\mathbb{R}^d$.
- Weight $w_{(u,v)} = |u-v|^\beta$, $\beta > 0$.

$$\Longrightarrow \boxed{e = (u,v) \in G_{n+1}\backslash G_n \quad \text{if} \quad \min(|C_{G_n}(u)|, |C_{G_n}(v)|) > |u-v|^\beta.}$$

Under $\mathrm{P}^0_\lambda = \mathrm{P}_\lambda * \delta_0$, define

$$\lambda_c(d,\beta) = \sup\left\{\lambda > 0 : \mathrm{P}^0_\lambda(\lim_n |C_{G_n}(0)| = \infty) = 0\right\}.$$

By monotonicity, we have:

- if $\lambda < \lambda_c$, $\mathrm{P}^0_\lambda(\lim_n |C_{G_n}(0)| = \infty) = 0$,
- if $\lambda > \lambda_c$, $\mathrm{P}^0_\lambda(\lim_n |C_{G_n}(0)| = \infty) > 0$.

PERCOLATION IN THE POISSON MODEL

Let $\lambda_{perc}(d)$ be the critical percolation threshold in the Gilbert's disc model with diameter 1 in dimension d.

If $\lambda > \lambda_{perc}(d)$ then $\mathrm{P}^0_\lambda(|C_{G_1}(0)| = \infty) > 0 \quad \Longrightarrow \quad 0 \leq \lambda_c(d) \leq \lambda_{perc}(d)$.

Natural percolation question:

Do we have $0 < \lambda_c(d, \beta) < \lambda_{perc}(d)$?

PERCOLATION IN DIMENSION ONE

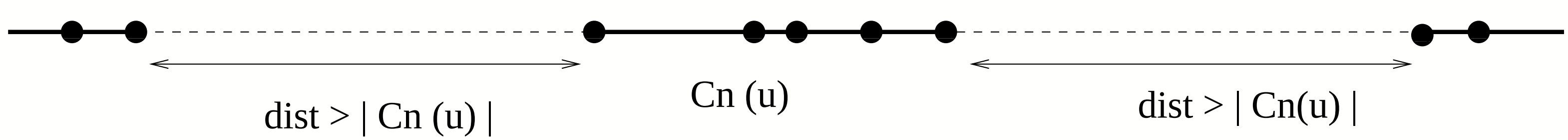
If $d = 1$, $\lambda_{perc}(d = 1) = +\infty$. For $\beta = 1$, the answer is yes.

Theorem 1. *For* $d = 1$ *in the Poisson model,*

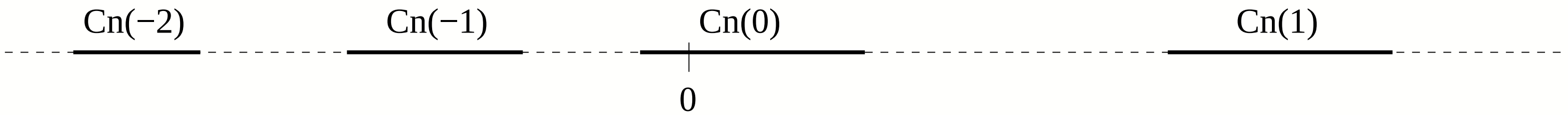
$$0 < \lambda_c(d = 1, \beta = 1) < +\infty.$$

IDEA OF PROOF: ANALYSIS OF THE GS RECURSION ($\beta = 1$)

For any vertex u, $(C_{G_n}(u))_{n\in\mathbb{N}}$ is increasing for $n \leq n(u)$ and constant for $n > n(u)$.



$\Longrightarrow$ we define $(C_n^i)_{i\in\mathbb{Z}}$ as the set of clusters still growing at time n.



IDEA OF PROOF: $\lambda_c < \infty$

We define $X_n^i = |C_n^i|$, $L_n^i = \mathrm{lenght}(C_n^i)$ and $Y_n^i = \mathrm{distance}(C_n^i, C_n^{i+1})$.

Sufficient to prove that for λ large enough,

$$\mathrm{P}_\lambda^0\left(\bigcap_{n\geq 0}\min(X_n^0, X_n^1) \geq Y_n^0\right) > 0.$$

Caveats

- The process $(X_n^i, L_n^i, Y_n^i), i \in \mathbb{N}$, is NOT an iid sequence,
- $\mathcal{L}(X_n^0, L_n^0, Y_n^0), n \in \mathbb{N}$, may NOT be a monotone sequence in law.

$\Longrightarrow$ We need to define a new recursion $(\widetilde{C}_n^i)_{i,n}$, easier to analyze and such that:

$$\forall n \in \mathbb{N}, \quad \widetilde{C}_{G_n}(u) \subseteq C_{G_n}(u)$$

IDEA OF PROOF: A NEW RECURSION

For an ordered sequence of sets $(M^i)_{i\in\mathbb{N}}$, define $X^i = |M^i|$, $L^i = \sup_{x,y\in C^i} |x-y|$, $Y^i = d(M^i, M^{i+1})$. Let

$$\tau^0 = \inf\{i \geq 0 : \min(X^i, X^{i+1}) > Y^i\},$$

and $\tau^{j+1} = \inf\{i \geq \tau^j + 2 : \min(X^i, X^{i+1}) > Y^i\}$.

The recursion maps $(M^i)_{i\in\mathbb{N}}$ to $(M'^i)_{i\in\mathbb{N}}$ with

$$M'^i = M^{\tau^i} \cup M^{\tau^i+1}.$$

$$\Longrightarrow \qquad X'^i = X^{\tau^i} + X^{\tau^i+1},$$

$$L'^i = L^{\tau^i} + L^{\tau^i+1} + Y^{\tau^i},$$

$$Y'^i = Y^{\tau^i+1} + \sum_{j=\tau^i+2}^{\tau^{i+1}-1} Y^j + L^j.$$

IDEA OF PROOF: NEW RECURSION

Initial condition, $\widetilde{C}_0^i = C_0^i = \{\eta_i\}_{i\in\mathbb{Z}}$, new sequence of sets $(\widetilde{C}_n^i)_{i\in\mathbb{N}}$.

For all $n \in \mathbb{N}$,

- $\widetilde{C}_{G_n}(u) \subseteq C_{G_n}(u)$,
- $\widetilde{X}_i^n = 2^n$,
- the sequence $(\widetilde{X}_n^i, \widetilde{L}_n^i, \widetilde{Y}_n^i)_{i\in\mathbb{N}}$ is iid,
- $(\widetilde{L}_n^i)_{i\in\mathbb{N}}$ and $(\widetilde{Y}_n^i)_{i\in\mathbb{N}}$ are independent.

IDEA OF PROOF: LARGE DEVIATION ESTIMATE

We need to prove that with positive probability at all stage n, the cluster $\widetilde{C}_n^0$ merges with its right neighbor. Sufficient to check that

$$\sum_{n \in \mathbb{N}} \mathrm{P}_\lambda^0(\widetilde{Y}_n > 2^n) < \infty.$$

Indeed, for λ large enough: $\widetilde{Y}_n = O((1+\varepsilon)^n)$.

Lemma 1. *Let* $0 < \varepsilon < 1$,

- *There exists* $c_0 > 0$ *such that if* $\mathrm{E}_\lambda^0 \exp(c_0 \widetilde{Y}_0) \le e$ *and* $\mathrm{E}_\lambda^0 \exp(c_0 (1+\varepsilon)^{-1} \widetilde{Y}_1) \le e$, *then for all* $n \ge 0$,

$$\mathrm{E}_\lambda^0 \exp(c_0 (1+\varepsilon)^{-n} \widetilde{Y}_n) \le e.$$

- *For all* $c > 0$,

$$\lim_{\lambda \to \infty} \mathrm{E}_\lambda^0 \exp(c \widetilde{Y}_0) = \lim_{\lambda \to \infty} \mathrm{E}_\lambda^0 \exp(c \widetilde{Y}_1) = 1.$$

$\longrightarrow$ proof by induction on $(\widetilde{Y}_n, \widetilde{L}_n)$ ...

A TEMPTING CONJECTURE

Recall,

$$J_G(u) = |C_G(u)| - \sum_{v \sim_G u} |u - v|^\beta.$$

Note that $\beta \mapsto \lambda_c(d, \beta)$ is non-decreasing. A scaling argument suggests

Conjecture 1. *For $d = 1$ in the Poisson model,*

(i) If $\beta > 1$ $\quad \lambda_c(1, \beta) = +\infty,$

(ii) If $0 < \beta < 1$ $\quad \lambda_c(1, \beta) = 0,$

(iii) If $\beta = 1$ $\quad 0 < \lambda_c(1, 1) < +\infty.$

$\longrightarrow$ no rigorous proof of (i) and (ii) available.

EXTENSION TO A CONCAVE SOCIAL CAPITAL

Let $0 < \alpha \leq 1$ and define:

$$J_G(u) = |C_G(u)|^\alpha - \sum_{v \sim_G u} |u - v|^\beta.$$

If $0 \leq \alpha < 1$, with the GS recursion, the sequence $(C_{G_n}(u))_{n \in \mathbb{N}}$ is no longer monotone: some connecting edges may be removed !

GREEDY, SELFISH BUT FAITHFUL RECURSION

Consider the greedy, selfish but faithful (GSF) recursion, where only the addition rule is used:

$$e = (u, v) \in G_{n+1} \backslash G_n \quad \text{if}$$

$$\min_{w \in \{u,v\}} (|C_{G_n}(u)| + |C_{G_n}(v)|)^\alpha - |C_{G_n}(w)|^\alpha > |u - v|^\beta.$$

Then monotonicity is again guaranteed. We may define the critical intensity $\lambda_c(d, \alpha, \beta)$. For $0 < \alpha \leq 1$, the same proof gives:

$$0 < \lambda_c(1, 1, 1) \leq \lambda_c(1, \alpha, \alpha) < +\infty.$$

MEAN FIELD MODEL OF DISTANCE

- The vertex set $V = [N] = \{1, \cdots, N\}$,
- Weight $w_{(u,v)} = \xi_{(u,v)}^{\beta}$, $\beta > 0$,
- Under the probability P_λ, $(\xi_e), e \in [N] \times [N]$ is an iid sequence of exponential variable with parameter λ/N.

$\Longrightarrow$ $(\xi_{(u,v)}^{1/d})_{v\in[N]}$ mimics the distances to the origin of a Poisson point process of intensity λ in $\mathbb{R}^d$.

$G_0^N = G_\emptyset^N$ and we consider the GSF recursion, and

$$e = (u,v) \in G_{n+1}^N \backslash G_n^N \quad \text{if}$$

$$\min_{w\in\{u,v\}} \left(|C_{G_n^N}(u)| + |C_{G_n^N}(v)|\right)^\alpha - |C_{G_n^N}(w)|^\alpha > \xi_e^\beta.$$

PERCOLATION IN THE MEAN FIELD MODEL

For all $n \geq \log_2(N)$, $G_n^N = G_*^N$

A percolation at intensity λ occurs if there exists $\rho = \rho(\lambda) > 0$ such that:

$$\liminf_{N\to\infty} \mathrm{P}_\lambda \left(|C_{G_*^N}(1)| \geq \rho N \right) > 0,$$

and we define:

$$\lambda_c(\infty, \alpha, \beta) = \sup \{\lambda > 0 : \text{percolation occurs at intensity } \lambda\} .$$

Do we have $0 < \lambda_c(\infty, \alpha, \beta) < \lambda_{perc}(\infty, \alpha, \beta) = (2^\alpha - 1)^{1/\beta}$?

PERCOLATION IN THE MEAN FIELD MODEL

The answer is no !

Theorem 2. *For all* $0 < \alpha \leq 1$, $\beta > 0$*, in the mean field model of distance:*

$$\lambda_c(\infty, \alpha, \beta) = 0$$

$\Longrightarrow$ Low price of anarchy in the mean field model of distance, irrespectively of the intensity of the model !

POISSON IN DIMENSION LARGER THAN ONE

For $d \geq 2$, the answer to the question

Do we have $0 < \lambda_c < \lambda_{perc}$?

is now less ”clear”...

POISSON IN DIMENSION LARGER THAN ONE

We have the following conjecture.

Let $\lambda_{perc}(d)$ be the critical percolation threshold in the Gilbert's disc model with diameter 1 in dimension d.

Conjecture 2. *In the Poisson model in dimension $d \geq 2$,*

(i) If $0 < \beta \leq d\alpha$ $\quad \lambda_c(d, \alpha, \beta) = 0,$

(ii) If $\beta > d\alpha$ $\quad 0 < \lambda_c(d, \alpha, \beta) < (2^\alpha - 1)^{d/\beta} \lambda_{perc}(d).$

FINAL COMMENTS

Beyond the conjecture, many other questions:

- In dimension $d \geq 2$, does there exists for all $\lambda > \lambda_c(d)$ an integer $n = n(\lambda)$ such that $\mathrm{P}^0_\lambda(|C_{G_n}(0)| = \infty) > 0$?
- Scaling of $n(\lambda)$ as λ goes down to $\lambda_c(d)$?
- More generally, all other classical questions of percolation (unicity of the infinite component, exponential decay, critical exponents)
- What about the GS recursion for $0 < \alpha < 1$?
- What about other random graphs models ?

\+ need of tractable mathematical models $\cdots$

On selfishness in networks, simulation study in a related model [Schneider and Kirkpatrick (2005)].