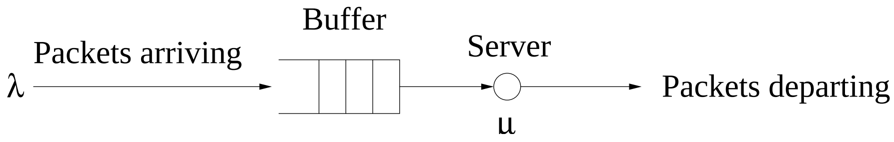
## $M/M/1$ Queue



Notation

- $A(t)$: number of packets that arrived in $[0, t]$
- $B(t)$: number of packets that departed in $[0, t]$
- $N(t) = A(t) - B(t)$: number of packets in the system (in queue and in service) at time $t$.

## Review: Poisson Process and Exponential Distribution

Time interval $[t, t+\delta]$

- $P\{A(t+\delta) - A(t) = 1\} =$
- $P\{A(t+\delta) - A(t) = 0\} =$
- $P\{B(t+\delta) - B(t) = 1\} =$
- $P\{B(t+\delta) - B(t) = 0\} =$

## Review

Time interval $[t, t+\delta]$

$$\begin{aligned}
P\{B(t+\delta) - B(t) = 1\} &= 1 - e^{-\mu\delta} \\
&= 1 - \Big(1 + \delta(-\mu e^{-\mu\delta} \mid_{\delta=0} + o(\delta)\Big) \\
&= 1 - \Big(1 + \delta(-\mu) + o(\delta)\Big) \\
&= \mu\delta + o(\delta))
\end{aligned}$$

Taylor expansion for a function $f(x) : \Re \to \Re$

$$f(x) = f(0) + x f'(0) + o(\delta)$$

## Transition Probabilities

For $n = 1, 2, \ldots$

$$\begin{aligned}
P\{N(t+\delta) = n+1 \mid N(t) = n\} &= \Big(\lambda\delta + o(\delta)\Big)\Big(1 - \mu\delta + o(\delta)\Big) \\
&= \lambda\delta + \lambda\mu\delta^2 + o(\delta)\Big(1 - \mu\delta + o(\delta)\Big) \\
&= \lambda\delta + o(\delta)
\end{aligned}$$

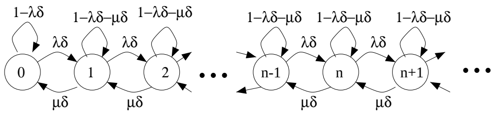
Note that

$$\begin{aligned}
&\lim_{\delta\to 0} \frac{\lambda\mu\delta^2}{\delta} = \delta = 0 \\
&\lim_{\delta\to 0} \frac{o(\delta)\mu\delta}{\delta} = o(\delta)\mu = 0 \\
&\lim_{\delta\to 0} o(\delta)^2 = 0.
\end{aligned}$$

## State Transition Diagram

Similarly, for $n = 1, 2, ...$

$$\begin{aligned} P\{N(t+\delta) = n-1 \mid N(t) = n\} &= \mu\delta + o(\delta) \\ P\{N(t+\delta) = n \mid N(t) = n\} &= 1 - \lambda\delta - \mu\delta + o(\delta) \end{aligned}$$



$$p_n = (1-\rho)\rho^n, \qquad n = 0, 1, 2, ....$$

for

$$\rho = \frac{\lambda}{\mu} < 1.$$

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## Next

- $N = \sum_{n=0}^{\infty} np_n = \frac{\rho}{1-\rho}$
- $T = \frac{1}{\mu-\lambda}$
- $P_{loss} \approx (1-\rho)\rho^B$
- As Internet changes - how does QoS change?
- Compare Packet-Switching with Circuit-Switching
- How should I upgrade my Network?

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## Average Number of Packets in the System

$$\begin{aligned} N &= \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n \\ &= (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\rho\sum_{n=0}^{\infty} n\rho^{n-1} \\ &= (1-\rho)\rho\frac{\partial}{\partial\rho}\left(\sum_{n=0}^{\infty}\rho^n\right) \\ &= (1-\rho)\rho\frac{\partial}{\partial\rho}\left(\frac{1}{1-\rho})\right) = (1-\rho)\rho\frac{1}{(1-\rho)^2} \\ &= \frac{\rho}{1-\rho} \end{aligned}$$

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## Average Packet Delay

Using Little's Theorem, we have

$$T = \frac{1}{\lambda}N,$$

or

$$\begin{aligned} T &= \frac{1}{\lambda}\frac{\rho}{1-\rho} \\ &= \frac{1/\mu}{1-\rho} \\ &= \frac{1}{\mu-\lambda} \end{aligned}$$

## Packet Loss

When the system can hold $B$ packets (1 packet in service and $(B-1)$ packets in the buffer), then we approximate

$$P_{loss} \approx (1-\rho)\rho^B.$$

You will show this in the next assignment.