Faculty of Arts and Science
University of Toronto
CSC 358 - Introduction to Computer Networks

Solutions Tutorial 5

Question 1
Consider a $M/M/1$ queue, with a packet arrival rate λ and service rate μ such that $\rho = \frac{\lambda}{\mu} < 1$.

(a) Assume that a new packet joins the queue and finds already n packets in the system (either waiting in the queue or in service). As a function of n, $n = 0, 1, 2, ...$, what is the expected waiting time of the new packet?

Recall that under the discrete-time model that we use, the service time of a packet is geometrically distributed, i.e. the probability that the service times is equal to k time slots of lenght δ is equal to

$$(1 - \mu\delta)^{k-1}\mu\delta, \qquad k = 1, 2, ...$$

Therefore, the expected number of time slots of serivce of a packet is equal to

$$\frac{1}{\mu\delta}\delta = \frac{1}{\mu}.$$

Let K be the random variable, indicating the number of packets that are already in the system when the new packet arrives. Note that the new packet has to wait until the K packets finish their service (queueing delay) and then has to finish its own service until it leaves the system (transmission delay). As the expected service time of a packet is equal to $\frac{1}{\mu}$, we have

$$E[T \mid K = n] = \frac{1}{\mu}(n + 1).$$

(b) For a $M/M/1$, the steady-state probability that n packets are in the system is equal to

$$p_n = (1 - \rho)\rho^n, \qquad n = 0, 1, 2, ..., .$$

Using the above equation and the results of (a), compute the expected waiting time of a packet.

We have that

$$E[T] = \sum_{n=0}^{\infty} E[T \mid K = n]p_n = \sum_{n=0}^{\infty} \frac{1}{\mu}(n + 1)p_n,$$

or

$$E[T] = \frac{1}{\mu} + \frac{1}{\mu}\sum_{n=0}^{\infty} np_n = \frac{1}{\mu} + \frac{1}{\mu}N.$$

Using the result that

$$N = \frac{\rho}{1-\rho},$$

we obtain

$$E[T] = \frac{1}{\mu}\Big(1 + \frac{\rho}{1-\rho}\Big) = \frac{1}{\mu}\frac{1}{1-\rho} = \frac{1}{\mu - \lambda}.$$

Question 2

Consider the $M/M/1$ that we discussed in class and let p_n, $n = 0, 1, 2, ...$, be the steady-state probability that n packets are in the system. Furthermore, let A be the event that less than n_0 packets are in the system, and let B be the event that the number of packets in the system is equal, or larger, than n_0.

(a) What is the probability of event A in steady-state, i.e. express $P\{A\}$ in terms of p_n, $n = 0, 1, 2, ...$

$$P\{A\} = \sum_{n=0}^{n_0-1} p_n.$$

(b) What is the probability of event B in steady-state, i.e. express $P\{B\}$ in terms of p_n, $n = 0, 1, 2, ...$

$$P\{B\} = \sum_{n=n_0}^{\infty} p_n = 1 - P\{A\}.$$

(c) Let $p_n(t)$ be the probability that n packets are in the system at time t, and let $A(t)$ be the event that at time t less than n_0 packets are in the system. For a very small δ, compute the probability $P\{A(t+\delta)\}$ as a function of $p_n(t)$, $n = 0, 1, 2, ...$, and $P\{A(t)\}$.

Using conditional probabilities, we obtain

$$\begin{aligned} P\{A(t+\delta)\} &= P\{A(t+\delta) \mid A(t)\}P\{A(t)\} + P\{A(t+\delta) \mid B(t)\}P\{B(t)\} \\ &= \big(1 - P\{B(t+\delta) \mid A(t)\}\big)P\{A(t)\} + P\{A(t+\delta) \mid B(t)\}P\{B(t)\} \end{aligned}$$

Note that

$$P\{B(t+\delta) \mid A(t)\} = \frac{P\{B(t+\delta) \cap A(t)\}}{P\{A(t)\}}$$

and

$$P\{B(t+\delta) \cap A(t)\} = p_{n_0-1}(t)\lambda\delta.$$

Similarly, we have that

$$P\{A(t+\delta) \mid B(t)\} = \frac{P\{A(t+\delta) \cap B(t)\}}{P\{B(t)\}}$$

and

$$P\{A(t+\delta) \cap B(t)\} = p_{n_0}(t)\mu\delta.$$

Using these results, it follows that

$$P\{A(t+\delta)\} = P\{A(t)\} - p_{n_0-1}(t)\lambda\delta + p_{n_0}(t)\mu\delta.$$

(d) Derive a condition on the probabilities $p_n(t)$, $n = 0, 1, 2, ...$, which implies that $P\{A(t+\delta)\} = P\{A(t)\}$ The condition is that

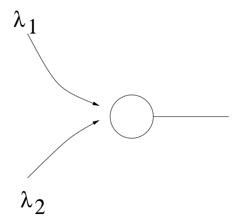
$$p_{n_0-1}(t)\lambda\delta = p_{n_0}(t)\mu\delta.$$

(e) How can we use the result of (d) to derive the steady-state probabilities p_n, $n = 0, 1, 2, ...$? We used the above condition to derive that in steady-state we have that

$$p_n\lambda\delta = p_{n+1}\mu\delta, \qquad n = 0, 1, 2, ...$$

Question 3

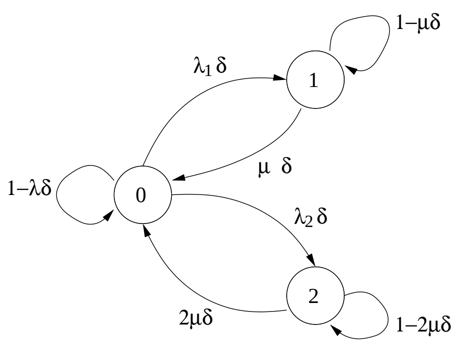
Consider a transmission system (queue) that can hold at most one packet (the packet that is in service), *i.e.* there is no buffer and a new packet either goes directly into service or is dropped.
The system receives Poisson packet traffic from two other nodes, 1 and 2, at rates λ_1 and λ_2, respectively. The service times of the packets are independently, exponentially distributed with a mean $\frac{1}{\mu}$ for packets from node 1, and $\frac{1}{2\mu}$ for packets from source 2.



(a) What is the probability that a packet that gets accepted into service is a packet from node 1?

$$\frac{\lambda_1}{\lambda_2 + \lambda_1}$$

(b) Compute the steady-state probabilities P_1, and P_2, that the system serves a packet from node 1, and node 2, respectively. Let the state 0,1, and 2, indicate that the



case that we find in the system no packet, one packet of source 1, and one packet of source 2, respectively.
The above figure then gives the state transition diagram, where $\lambda = \lambda_1 + \lambda_2$. If follows that

$$P_0\lambda_1\delta = \mu\delta P_1$$
$$P_0\lambda_2\delta = 2\mu\delta P_2,$$

or

$$P_1 = \frac{\lambda_1}{\mu} P_0$$
$$P_2 = \frac{\lambda_2}{2\mu} P_0.$$

Using the condition that

$$P_0 + P_1 + P_2 = 1,$$

we obtain that

$$P_0\Big(1 + \frac{\lambda_1}{\mu} + \frac{\lambda_2}{2\mu}\Big) = 1.$$

Setting

$$\bar{\rho} = \frac{2\lambda_1 + \lambda_2}{2\mu},$$

we get

$$P_0 = \frac{1}{1 + \bar{\rho}},$$

and

$$P_1 = \rho_1 \frac{1}{1 + \bar{\rho}}$$
$$P_2 = \rho_2 \frac{1}{1 + \bar{\rho}}$$

where

$$\rho_1 = \frac{\lambda_1}{\mu}$$

and

$$\rho_2 = \frac{\lambda_2}{2\mu}.$$