Faculty of Arts and Science

University of Toronto

CSC 358 - Introduction to Computer Networks

Solutions for Tutorial 2

Topic

In the course, we will use probabilistic models for packet lengths and packet arrivals. The goal of this tutorial is to get familiar with these models.

Question 1: Packet Length

The length of data packets can vary in a wide range (some packets are very short and some packets are very long). To capture this, we model packet lengths as a random variable with a geometric distribution. That is, the probability that a packet is L bits long is given by.

$$P(L = l) = \mu(1-\mu)^{l-1}, \qquad l \geq 1.$$

(a) Derive the average packet length, i.e. derive $E[L]$. We then have

$$\begin{aligned}
E[L] &= \sum_{l=1}^{\infty} l\mu(1-\mu)^{l-1} \\
&= \mu \frac{d}{d\mu} \sum_{l=1}^{\infty} \left[(-1)(1-\mu)^{l}\right] \\
&= \mu \frac{d}{d\mu} \left[(\mu - 1) \sum_{l=1}^{\infty} (1-\mu)^{l-1}\right] \\
&= \mu \frac{d}{d\mu} \left[(\mu - 1) \sum_{l=0}^{\infty} (1-\mu)^{l}\right] \\
&= \mu \frac{d}{d\mu} \left[(\mu - 1) \frac{1}{\mu}\right] = \mu \frac{d}{d\mu} \left[1 - \frac{1}{\mu}\right] \\
&= \mu \frac{1}{\mu^2} \\
&= \frac{1}{\mu}.
\end{aligned}$$

(b) Consider a specific packet. Assume that we know that the length of this packet is larger than l_0. Find the probability that the packet is l bits long, $l > l_0$.

For $l > l_0$, we have that

$$P(L = l | L > l_0) = \frac{P(L = l \text{ and } L > l_0)}{P(L > l_0)} = \frac{P(L = l)}{P(L > l_0)}.$$

Why is this the case?

Note that

$$P(L = l) = \mu(1 - \mu)^{l-1},$$

and

$$P(L > l_0) = \sum_{l=l_0}^{\infty} \mu(1 - \mu)^l = (1 - \mu)^{l_0} \sum_{l=0}^{\infty} \mu(1 - \mu)^l = (1 - \mu)^{l_0}.$$

Why is this the case?

In then follows that for $l > l_0$, we have

$$P(L = l | L > l_0) = \frac{P(L = l)}{P(L > l_0)} = \frac{\mu(1 - \mu)^{l-1}}{(1 - \mu)^{l_0}} = \mu(1 - \mu)^{l-l_0-1}.$$

Note that this implies that the geometric distribution is "memoryless". What does this mean?

(c) Derive the expected packet length when we know that $L > l_0$, i.e. derive $E[L \mid L > l_0]$.

Using part (b), we have that

$$\begin{aligned}
E[L \mid L > l_0] &= \sum_{l=l_0+1}^{\infty} l\mu(1 - \mu)^{l-l_0-1} \\
&= \sum_{l=l_0+1}^{\infty} (l - l_0 + l_0)\mu(1 - \mu)^{l-l_0-1} \\
&= \sum_{x=1}^{\infty} (x + l_0)\mu(1 - \mu)^{x-1} \\
&= \sum_{x=1}^{\infty} x\mu(1 - \mu)^{x-1} + \sum_{x=1}^{\infty} l_0\mu(1 - \mu)^{x-1} \\
&= \sum_{x=1}^{\infty} x\mu(1 - \mu)^{x-1} + l_0 \sum_{x=1}^{\infty} \mu(1 - \mu)^{x-1}
\end{aligned}$$

Using the result from (b), we then have that

$$E[L \mid L > l_0] = \frac{1}{\mu} + l_0.$$

What is the interpretation of this result?

Question 2: Packet Arrivals

In the course, we will use the following discrete-time model to characterize packet arrivals.

Suppose that time is divided into slots of length Δ_t and consider the following packet arrival process with rate λ:

1. the probability of one packet arriving during a time-slot is equal to $\lambda\Delta_t$.
2. the probability of zero arrival in the interval Δ_t is $1 - \lambda\Delta_t$.
3. arrivals are memoryless: An arrival (event) in one time interval of length Δ_t is independent of events in previous intervals.

Using this model, answer the following questions.

(a) Consider a time interval of length $k\Delta_t$. What is the probability that we have n arrivals in the time interval $[0, k\Delta_t]$ for $n = 0, ..., k$?

For $0 \leq n \leq k$, the probability that we have n arrivals in the time interval $[0, k\Delta_t]$ is given

$$\binom{k}{n} (\lambda\Delta_t)^n (1 - \lambda\Delta_t)^{k-n},$$

which corresponds to a binomial distribution.

(b) What is the distribution of the time between two successive packet arrivals (inter-arrival time)?

Let T be the inter-arrival time, then we have that

$$P(T = t) = (\lambda\Delta_t)(1 - \lambda\Delta_t)^{t-1}, \qquad t \geq 1,$$

i.e. the inter-arrival time is given by a geometric distribution. why is this the case?

Question 3: Poisson and Exponential Distribution

In this question, we show what happens for the model of Question 2 as we make Δ_t smaller and smaller, letting it approach 0.

Consider a time interval of fixed length T which is divided into N slots of equal length $\Delta_t = T/N$. In each time slot, exactly one new packet arrives with probability $\lambda\Delta_t$, and no packet arrives with probability $1 - \lambda\Delta_t$. The probability that two more packets arrive is equal to 0.

(a) What is the probability P_n that n, $n = 0, 1, ..., N$, packets arrive in the time interval $[0, T]$.

$$P_n = \binom{N}{n}(\lambda\Delta_t)^n(1-\lambda\Delta_t)^{N-n}, \qquad n = 0, ..., N.$$

(b) Find the probability P_n as the number of time slots N approaches infinity ($N \to \infty$) (and the interval Δ_t approaches 0, $\Delta_t \to 0$). Hint: Use $\lim_{x\to 0}(1+ax)^{\frac{k}{x}} = e^{ak}$ and for N very large, $N! \approx \frac{(N/e)^N}{\sqrt{2N\pi}}$ (Stirling's approximation).

We do the analysis without using Stirling's approximation.

$$\begin{aligned} P_n &= \binom{N}{n}(\lambda\Delta_t)^n(1-\lambda\Delta_t)^{N-n} \\ &= \frac{N!}{(N-n)!n!}\left(\lambda\frac{T}{N}\right)^n(1-\lambda\Delta_t)^{T/\Delta_t}(1-\lambda\Delta_t)^{-n} \\ &= \frac{N(N-1)\dots(N-(n-1))}{N^n}\frac{(\lambda T)^n}{n!}(1-\lambda\Delta_t)^{T/\Delta_t}(1-\lambda\Delta_t)^{-n} \end{aligned}$$

For T fixed, as $N \to \infty$, we have $\Delta_t = \frac{T}{N} \to 0$,

and

- $\lim_{\Delta_t\to 0}(1-\lambda\Delta_t)^{-n} = 1$
- $\lim_{\Delta_t\to 0}(1-\lambda\Delta_t)^{T/\Delta_t} = e^{-\lambda T}$ (Equation for the exponential)
- $\lim_{N\to\infty}\frac{N(N-1)...(N-(n-1))}{N^n} = \lim_{N\to\infty} 1(1-\frac{1}{N})(1-\frac{2}{N})\dots(1-\frac{n-1}{N}) = 1$

We then obtain that

$$P_n = \frac{(\lambda T)^n}{n!}e^{-\lambda T}, \qquad n = 0, 1, 2, ...$$

which is the Poisson distribution.

(c) Assuming that $\Delta_t \to 0$, what is the distribution of the time between two successive packet arrivals?

Let T be the inter-arrival time. Then using (b), we have that

$$\begin{aligned} P(T \leq t) &= P(\text{ at least one arrival in the interval [0,t)}) \\ &= 1 - P(\text{ no arrivals in the interval [0,t)}) \\ &= 1 - e^{-\lambda t}, \qquad t > 0, \end{aligned}$$

i.e. the inter-arrival time is given by an exponential distribution.