

Performance Analysis

- Performance measure $R(t)$
- How to characterize "system performance"?
 - Time Average:** "I observe the system for a long, long time".
 - Ensemble Average:** "I come back at time t to check on the system"
- $R_t = \frac{1}{t} \int_0^t R(\tau) d\tau$: time average up to time t .
- $R = \lim_{t \rightarrow \infty} R_t$: time average
- $\bar{R}(t) = \sum_{n=0}^{\infty} n p_n(t)$: expected number of jobs at time t
- $\bar{R} = \lim_{t \rightarrow \infty} \bar{R}(t)$: steady-state expected number of jobs in the system

Ergodic System: $R = \bar{R}$

1

How to Computer R and \bar{R} ?

Assumption: $E[X] < \infty$

Result: $R = \bar{R} = \frac{E[R_n]}{E[X]}$ (under some conditions)

Approach

- Renewal Process
- Renewal Reward Processes

2

Renewal Process

Assumptions: X_1, X_2, \dots positive IID r.v.; $\bar{X} < \infty$,

Results: (Note that $0 < \bar{X}$)

$$\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}$$

$$\lim_{t \rightarrow \infty} \frac{E[N(t)]}{t} = \frac{1}{\bar{X}}$$

$$\lim_{t \rightarrow \infty} (E[N(t + \delta)] - E[N(t)]) = \frac{\delta}{\bar{X}}, \quad \delta > 0$$

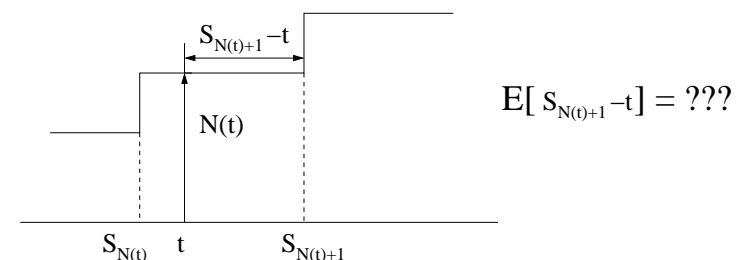
3

$$m(t) = E[N(t)]$$

Using Wald's equality:

- $m(t) \geq \frac{1}{E[X]} - \frac{1}{t}$
- $m(t) \leq \tilde{m}(t) \leq \frac{1}{E[X]} + \frac{b_t}{tE[X]} - \frac{1}{t}$,
where we used

$$\{\tilde{m}(t); t > 0\}, \quad b_t = \sqrt{t}.$$



4

Blackwell's Theorem

Arithmetic Distribution: Renewals only happen at integer multiples of some real number $d > 0$.

Span of an Arithmetic Distribution: The largest real number d such that the above property holds

5

Blackwell's Theorem

- For non-arithmetic inter-renewal distribution and $\delta > 0$:

$$\lim_{t \rightarrow \infty} m(t + \delta) - m(t) = \delta \frac{1}{E[X]}$$

- For arithmetic inter-renewal distribution with span d and any **integer** $n \geq 1$

$$\lim_{t \rightarrow \infty} m(t + nd) - m(t) = nd \frac{1}{E[X]}$$

- For non-arithmetic inter-renewal distribution and $\delta > 0$:

$$\lim_{t \rightarrow \infty} P(N(t + \delta) - N(t) = 0) = 1 - \delta \frac{1}{E[X]} + o(\delta)$$

$$\lim_{t \rightarrow \infty} P(N(t + \delta) - N(t) = 1) = \delta \frac{1}{E[X]} + o(\delta)$$

$$\lim_{t \rightarrow \infty} P(N(t + \delta) - N(t) \geq 2) = o(\delta)$$

Note: Similar to Poisson process, but increments are **not necessarily independent**.

6

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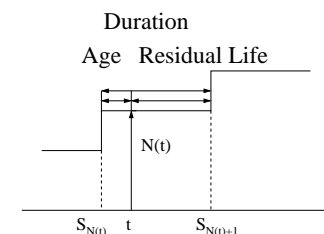
7

Renewal Reward Process

$\{N(t); t \geq 0\}$: Renewal Process

$\{R(t); t \geq 0\}$: Reward Function (need some assumptions)

- $R_n = \int_{S_{n-1}}^{S_n} R(t) dt$: IID r.v.
- $R = \bar{R} = \frac{E[R_n]}{E[X]}$ (non-arithmetic)



8