CSC2206 Tutorial 2 - Exercise

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Proposition 1. *Let Y be a non-negative random variable with finite mean, show that*

$$\lim_{y\to\infty} y\,P(Y \geq y) = 0 \tag{1}$$

The idea of the proof is to first give an upper bound to $y\,P(Y \geq y)$ and then show that the upper bound converges to 0 as $y \to \infty$. More precisely, we will prove the following two lemmas.

Lemma 1. *Let Y be a random variable with distribution function F_Y, then*

$$y\,P(Y \geq y) \leq \int_{z=y}^{\infty} z\,dF_Y(z) \tag{2}$$

Lemma 2. *Let Y be a non-negative random variable with finite mean, then*

$$\lim_{y\to\infty} \int_{z=y}^{\infty} z\,dF_Y(z) = 0 \tag{3}$$

Proof of Lemma 1:

$$\begin{aligned} y\,P(Y \geq y) &= y\,(1 - F_Y(y)) & (4)\\ &= y\int_{z=y}^{\infty} dF_Y(z) & (5)\\ &= \int_{z=y}^{\infty} y\,dF_Y(z) & (6)\\ &\leq \int_{z=y}^{\infty} z\,dF_Y(z) & (7) \end{aligned}$$

Equality (5) follows from that $F_Y(\infty) = 1$. Inequality (7) holds because $z \geq y$. ■

Proof of Lemma 2:

$$\lim_{y\to\infty} \int_{z=y}^{\infty} z\,dF_Y(z) = \lim_{y\to\infty} \left(\int_{z=0}^{\infty} z\,dF_Y(z) - \int_{z=0}^{y} z\,dF_Y(z) \right) \tag{8}$$

$$= \lim_{y\to\infty} \left(E[Y] - \int_{z=0}^{y} z\, dF_Y(z) \right) \tag{9}$$

$$= E[Y] - \lim_{y\to\infty} \int_{z=0}^{y} z\, dF_Y(z) \tag{10}$$

$$= E[Y] - E[Y] \tag{11}$$

$$= 0 \tag{12}$$

Equalities (9) and (11) follow from the assumption that Y is non-negative. Equality (12) is true because $E[Y] < \infty$. ■

Proof of Proposition 1: By Lemma 1 and 2,

$$\lim_{y\to\infty} y\, P(Y \geq y) \leq \lim_{y\to\infty} \int_{z=y}^{\infty} z\, dF_Y(z) = 0 \tag{13}$$

On the other hand, clearly

$$\lim_{y\to\infty} y\, P(Y \geq y) \geq 0 \tag{14}$$

Therefore,

$$\lim_{y\to\infty} y\, P(Y \geq y) = 0 \tag{15}$$

■