CSC2206 Tutorial 2

Larry Zhang

I. Basic Concepts

Proposition 1. *Let X be a random variable, it is finite with probability 1, but it does NOT necessarily have finite mean,* i.e.,

$$P(X < \infty) = 1 \qquad \textit{is true}$$

$$E[X] < \infty \qquad \textit{is not always true}$$

Below is an example of a random variable with infinite mean. Consider X with the following distribution function:

$$P(X \le x) = \begin{cases} 1 - \frac{1}{x} & \text{if } x \ge 1 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2. *(Union bound) For a countable set of events $A_1, A_2, A_3, \ldots$, we have*

$$P\left(\bigcup_i A_i\right) \le \sum_i P(A_i)$$

II. Convergence of Random Variables

Definition 1. *(Convergence of real numbers) Given a sequence $\{a_n\}$ of real numbers, we say*

$$\lim_{n\to\infty} a_n = a$$

if $\forall \epsilon > 0$, $\exists n_0 > 0$ *such that* $|a_n - a| < \epsilon, \forall n > n_0$.

Definition 2. *(Convergence in probability) Given a sequence of $\{X_n\}$ of random variables, we say the random sequence converges to a real number b in probability ($X \xrightarrow{p} b$), if $\forall \epsilon > 0$*

$$\lim_{n\to\infty} P(|X_n - b| > \epsilon) = 0$$

Definition 3. *(Convergence with probability 1) Given a sequence of* $\{X_n\}$ *of random variables, we say the random sequence converges to a real number* b *with probability 1 (or almost surely,* $X \xrightarrow{a.s.} b$*), if* $\forall \epsilon > 0$

$$\lim_{n\to\infty} P\left(\sup_{m\geq n} |X_m - b| > \epsilon\right) = 0$$

or equivalently,

$$\lim_{n\to\infty} X_n = b \quad \textit{w. p. 1}$$

For an example of a sequence that converge in probability but not almost surely, informally, think about the following sequence

$$1010010001000010000001\ldots$$

Formally, consider the sequence $\{X_n\}$ with the following distribution

$$X_n = \begin{cases} 1 & w.p. \ \frac{1}{n} \\ 0 & w.p. \ 1 - \frac{1}{n} \end{cases}$$

Claim 1. $X_n \xrightarrow{p} 0$ *but not* $X_n \xrightarrow{a.s.} 0$

Proof: First, for convergence in probability, we want to show the following

$$\lim_{n\to\infty} P(|X_n| > \epsilon) = 0, \forall \epsilon > 0$$

If $\epsilon \geq 1$, it is trivially true since X_n can only be 0 or 1.

If $0 < \epsilon < 1$, we have

$$\lim_{n\to\infty} P(|X_n| > \epsilon) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Second, to disprove almost surely convergence, we want to show that, $\forall 0 < \epsilon < 1$

$$\lim_{n\to\infty} P\left(\sup_{m\geq n} |X_m| > \epsilon\right) = 1$$

and is equivalent to

$$\lim_{n\to\infty} P\left(\sup_{m\geq n} |X_m| \leq \epsilon\right) = 0$$

We have

$$P\left(\sup_{m\geq n} |X_m| \leq \epsilon\right) = P\left(\bigcap_{m\geq n} X_m = 0\right) = \prod_{m\geq n} \left(1 - \frac{1}{m}\right)$$

To show that above product is 0, it suffices to show that

$$\log \prod_{m \geq n} \left(1 - \frac{1}{m}\right) = -\infty$$

which is true because

$$\log \prod_{m \geq n} \left(1 - \frac{1}{m}\right) = \sum_{m \geq n} \log(1 - \frac{1}{m}) \leq \sum_{m \geq n} -\frac{1}{m} = -\infty$$

where the inequality comes from the following inequality obtained using Tylor expansion

$$\log(1 - x) = -x - \frac{1}{2}x^2 - \cdots \leq -x, \quad \forall\, 0 \leq x < 1$$

$\blacksquare$