Final

- Dec. 8, 2-5pm, GB304
- Closed book
- Entire course
- Proofs: Chapter 3 and 5

Renewal Process

Assumptions: $X_1, X_2, ...$ positive IID r.v.;$\bar{X} < \infty$,

Results: (note that $0 < \bar{X}$)

$lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}, \qquad w.p.1$

$lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\bar{X}}$

$lim_{t\to\infty} \left(E[N(t+\delta)] - E[N(t)]\right) = \frac{\delta}{\bar{X}}, \quad \delta > 0$; (non-arithmetic)

Renewal Reward Process

- If $\bar{X} < \infty$ or $E[R_n] < \infty$, then
$$\lim_{t\to\infty} \frac{1}{t}\int_0^t R(\tau)d\tau = \frac{E[R_n]}{\bar{X}}, \qquad \text{with probability 1.}$$
- If non-arithmetic renewal process and $r(z)$ is directly Rieman integrable, then
$$\lim_{t\to\infty} E[R(t)] = \frac{E[R_n]}{\bar{X}}.$$
- If arithemtic renewal process with span d, then
$$\lim_{n\to\infty} E[R(nd)] = \frac{E[R_n]}{\bar{X}}.$$

Finite State Markov Chain

- Questions
  - Does $\pi = \pi[P]$ have a probability vector solution?
  - Does $\pi = \pi[P]$ have a unique probability vector solution?
  - Is $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$

Finite State Markov Chain - Answers

- Yes, solution to $\pi = \pi[P]$ always exists
- Unique, if and only if there is a single recurrent class (and possibly many transient classes)
- If there are r recurrent classes, then there exist r linearly independent solutions.
- For ergodic Markov chain we have

$$\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$$

- If there are several multiple recurrent classes $\lim_{n\to\infty}[P]^n$ exists, but rows are not identical.
- If there is one or more periodic class then $[P]^n$ does not converge.

Finite State Markov Chain with Rewards

- Single Recurrent Class: $v(n) = nge + w + [P]^n\{v(0) - w\}$
- Ergodic: $\lim_{n\to\infty}\{v(n) - nge\} = w + \beta e;, \qquad \beta = \pi(v(0) - w)$

Markov Chains with Countably Infinite State Spaces

- T_{ij}: "first passage time from i to j"
  - $f_{ij}(n)$: probability mass function
  - $F_{ij}(n)$: probability distribution functionq
- Classification
  - **recurrent**: state i is recurrent if $F_{ii}(\infty) = 1$
    - **positive recurrent** $E[T_{ii}] < \infty$
    - **null recurrent** $E[T_{ii}] = \infty$
  - **transient**: state i is transient if $F_{ii}(\infty) < 1$

Renewal Theory

Assume that state j is recurrent and consider the renewal process $\{N_{jj}(t); t \geq 0\}$. Then

- $\lim_{t\to\infty} \frac{N_{jj}(t)}{t} = \frac{1}{E[T_{jj}]}, \qquad w.p.1$
- $\lim_{t\to\infty} \frac{E\left[N_{jj}(t)\right]}{t} = \frac{1}{E[T_{jj}]}$
- If state j is periodic with span d, then

$$\lim_{n\to\infty} P(X_{nd} = j \mid X_0 = j) = \frac{d}{\bar{T}_{jj}}$$

- If state j is aperiodic, then

$$\lim_{n\to\infty} P(X_n = j \mid X_0 = j) = \frac{1}{\bar{T}_{jj}}$$

Steady-State Distribution

Vector $(\pi_0, \pi_1, \pi_2, ...)$, $\pi_i \geq 0$, such that

$$\pi_i = \sum_j \pi_j P_{ji}$$

and

$$\sum_i \pi_i = 1$$

- **Theorem:** Consider an irreducible M.C. with transition probabilities $\{P_{ij}\}$. If the above equation has a solution, the the solution is **unique**, we have $\pi_i = \frac{1}{T_{ii}}$ for all $i \geq 0$, and all states are **positive recurrent**. Also, if all states are positive recurrent, then the above equation has a solution.

Semi-Markov Processes

- $X(t)$ state of process at time $t \geq 0$, $X(t) \in \{0, 1, 2, ...\}$
- $S_1 < S_2 < S_3 < ...$: epochs at which transitions occur
- X_n: new state entered at time n: $X_n = X(S_n)$, $X(t) = X_n$ for $S_n \leq t < S_{n+1}$
- $\{X_n : n \geq 0\}$ is a M.C. with $\{P_{ij}\}$. This M.C. is called the embedded M.C.
- $U_n = S_n - S_{n-1}$ is a R.V
  - depends only on X_{n-1}, X_n

$$P(U_n \leq u \mid X_{n-1} = i, X_n = j) = G_{ij}(u)$$

  - $\bar{U}(i, j)$: conditional mean of transition time
  - $\bar{U}(i) = \sum P_{ij} \bar{U}(i, j)$

Semi-Markov Processes

- **Lemma**: Consider a semi-Markov process with an **irreducible** recurrent embedded M.C. $\{X_n : n \geq 0\}$. Given $X_0 = i$, let $M_{ij}(t)$ be the number of transitions into a given state j in the interval $(0, t]$. Then $M_{ij}(t)$ is a delayed renewal process (and $M_{jj}(t)$ is a renewal process).
- **Theorem**: Assume that the embedded M.C. of a semi-Markov process is irreducible and positive recurrent. If $\sum_i \pi_i \bar{U}(i) < \infty$, then, with probability 1, the limiting fraction of time spent in state i is

$$p_i = \frac{\pi_i \bar{U}(i)}{\sum_j \pi_j \bar{U}(j)}.$$

Special Cases

- Markov chains with countably infinite state spaces
  - Branching processes
  - Birth-death Markov chains
  - Reversible Markov chains
- Semi-Markov processes
  - Markov processes
    - Birth-Death Markov processes
    - Reversible Markov

Dyanmic Programming

- Extension of finite state Markov chains with rewards
- Optimal stationary policy B
  - Single Recurrent Class:
    $v^B(n) = ng^B e + w^B + [P^B]^n\{v^B(0) - w^B\}$
  - Ergodic:
    $\lim_{n\to\infty}\{v^B(n) - ng^B e\} = w^B + \beta e;, \qquad \beta = \pi^B(v(0) - w^B)$