Semi-Markov Processes

- $X(t)$ state of process at time $t \geq 0$
- $X(t) \in \{0, 1, 2, ...\}$
- $S_1 < S_2 < S_3 < ...$: epochs at which transitions occur
- X_n: new state entered at time n: $X_n = X(S_n)$
- $X(t) = X_n$ for $S_n \leq t < S_{n+1}$
- $S_0 = 0$, X_0

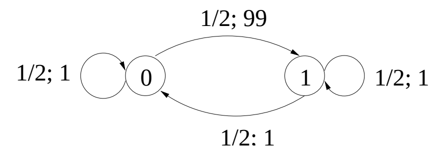
Semi-Markov Processes: Requirements

- $\{X_n : n \geq 0\}$ is a M.C. with $\{P_{ij}\}$. This M.C. is called the embedded M.C.
- $P_{ii} =$
- $U_n = S_n - S_{n-1}$
  - is a R.V
  - depends only on X_{n-1}, X_n

$$P(U_n \leq u \mid X_{n-1} = i, X_n = j) = G_{ij}(u)$$

  - $\bar{U}(i, j)$: conditional mean of transition time
  - $\bar{U}(i) = \sum P_{ij} \bar{U}(i, j)$
- Visualization

Semi-Markov Processes: Example



- Embedded M.C.: $\pi_0 = ...$, $\pi_1 = ...$
- Semi-Markov process: p_0, p_1
- Guess
  - $p_i \sim \pi_i \bar{U}(i)$
  - $\sum_j p_j = 1$
  - $p_i = \frac{\pi_i \bar{U}(i)}{\sum_j \pi_j \bar{u}(j)}$

Semi-Markov Processes: Analysis

- **Lemma**: Let $M(t)$ be the number of transitions in a semi-Markov process in the interval $(0, t]$ for some given initial state X_0. Then with probability 1 we have

$$\lim_{t \to \infty} M(t) = \infty.$$

- **Lemma**: Consider a semi-Markov process with an **irreducible** recurrent embedded M.C. $\{X_n : n \geq 0\}$. Given $X_0 = i$, let $M_{ij}(t)$ be the number of transitions into a given state j in the interval $(0, t]$. Then $M_{ij}(t)$ is a delayed renewal process (and $M_{jj}(t)$ is a renewal process).

Semi-Markov Processes: Analysis

- **Theorem**: Assume that the embedded M.C. of a semi-Markov process is irreducible and positive recurrent. If $\sum_i \pi_i \bar{U}(i) < \infty$, then, with probability 1, the limiting fraction of time spent in state i is

$$p_i = \frac{\pi_i \bar{U}(i)}{\sum_j \pi_j \bar{U}(j)}.$$

Semi-Markov Processes: Analysis

- Extensions
  - Sample average:

$$\lim_{t \to \infty} P(X(t) = i) = p_i$$

    if $G_{ij}(u)$ is non-arithmetic for at least one pair i, j of states such that $P_{ij} > 0$.
  - $Q(i, j)$: fraction of time spent in transition from state i to j

$$Q(i, j) = p_i \frac{P_{ij} \bar{U}(i, j)}{\bar{U}(i)}$$