Renewal Process

Assumptions: $X_1, X_2, ...$ positive IID r.v.;$\bar{X} < \infty$,

Results: (Note that $0 < \bar{X}$)

$lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}$

$lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\bar{X}}$

$lim_{t\to\infty} (E[N(t+\delta)] - E[N(t)]) = \frac{\delta}{\bar{X}}, \quad \delta > 0$

Renewal Reward Process

- If $\bar{X} < \infty$ or $E[R_n] < \infty$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(\tau)d\tau = \frac{E[R_n]}{\bar{X}}, \qquad \text{with probability 1.}$$

- If non-arithmetic renewal process and $r(z)$ is directly Rieman integrable, then

$$\lim_{t\to\infty} E[R(t)] = \frac{E[R_n]}{\bar{X}}.$$

- If arithemtic renewal process with span d, then

$$\lim_{n\to\infty} E[R(nd)] = \frac{E[R_n]}{\bar{X}}.$$

Finite State Markov Chain

- Questions
  - Does $\pi = \pi[P]$ have a probability vector solution?
  - Does $\pi = \pi[P]$ have a unique probability vector solution?
  - Is $\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$

Finite State Markov Chain - Answers

- Yes, solution to $\pi = \pi[P]$ always exists
- Unique, if and only if there is a single recurrent class (and possibly many transient classes)
- If there are r recurrent classes, then there exist r linearly independent solutions.
- For ergodic Markov chain we have

$$\lim_{n\to\infty}[P]^n = \begin{bmatrix} \pi_1 & \cdots & \pi_J \\ \vdots & & \vdots \\ \pi_1 & \cdots & \pi_J \end{bmatrix}$$

- If there are several multiple recurrent classes $\lim_{n\to\infty}[P]^n$ exists, but rows are not identical.
- If there is one or more periodic class then $[P]^n$ does not converge.

Finite State Markov Chain with Rewards

- Single Recurrent Class: $v(n) = nge + w + [P]^n\{v(0) - w\}$
- Ergodic: $\lim_{n\to\infty}\{v(n) - nge\} = w + \beta e; ,\qquad \beta = \pi(v(0) - w)$