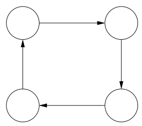
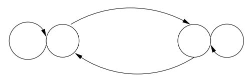
Reversibility

- Forward Chain: ..., $X_{n-1}, X_n, X_{n+1}, ...$
- Backward Chain: ..., $X_{n+1}, X_n, X_{n-1}, ...$





- Markov Chain $\{X_n; n \geq 0\}$

$$P(X_{n+k}, X_{n+k-1}, ..., X_{n+1}|X_n, X_{n-1}, ..., X_0) = P(X_{n+k}, X_{n+k-1}, ..., X_{n+1}|X_n)$$

- E^+: Event on $X_{n+1}, ..., X_{n+k}$
- E^-: Event on $X_0, ..., X_{n-1}$

Reversibility

- $P(E^+|X_n, E^-) = P(E^+|X_n)$
- $P(E^+|X_n, E^-)P(E^-|X_n) = P(E^+|X_n)P(E^-|X_n)$
- Conditional Independence:
  $P(E^+E^-|X_n) = P(E^+|X_n)P(E^-|X_n)$
- $P(E^-|X_n, E^+) = P(E^-|X_n)$
- Backward Chain is also a M.C.

$$P(X_{n-1}|X_n, X_{n+1}, ..., X_{n+k}) = P(X_{n-1}|X_n)$$

Backward Chain

- $P(X_{n-1}|X_n) = \frac{P(X_{n_1}X_n)}{P(X_n)} = \frac{P(X_n|X_{n-1})P(X_{n-1})}{P(X_n)}$
- Backward Chain is not a homogeneous M.C., but transition probabilities depend on n
- What about if M.C. is in steady state?

$$P(X_{n-1} = j|X_n = i) = \frac{P_{ji}\pi_j}{\pi_i} = P^*_{ij}$$

- $\pi_i P^*_{ij} = \pi_j P_{ji}$
- What if $P^*_{ij} = P_{ij}$

Reversibility

- A M.C. is **reversible** if $P^*_{ij} = P_{ij}$ and the forward and backward chain can not be distinguished
- **Theorem**: Every birth-death M.C. with a steady state probability distribution is reversible
- **Theorem**: Irreducible M.C. $\{X_n : n \geq 0\}$ and $\{P_{ij}\}$, $\{\pi_i\}$ such that $\pi_i > 0$ and $\sum \pi = 1$, and

$$\pi_i P_{ij} = \pi_j P_{ji}.$$

The $\{\pi_i\}$ is the steady state distribution and the chain is reversible.

Reversibility

- When is a M.C. **not** reversible?
  - If span d is larger than 2
  - If $P_{ij} > 0$ but $P_{ji} = 0$ for some i, j
  - more tests ....

Backward Chain

- **Theorem**: Irreducible M.C. $\{P_{ij}\}$ and $\{P^*_{ij}\}$, $\{\pi_i\}$ such that $\pi_i > 0$ and $\sum \pi = 1$, and

  $$\pi_i P_{ij} = \pi_j P^*_{ji}.$$

  The $\{\pi_i\}$ is the steady state distribution and $\{P^*_{ij}\}$ is the set of transition probabilities for the backward chain.