Performance Analysis

- Performance measure $R(t)$
- How to characterize "system performance"?

  Time Average: "I observe the system for a long, long time".

  Ensemble Average: "I come back at time t to check on the system "
- $R_t = \frac{1}{t}\int_0^t R(\tau)d\tau$: time average up to time t.
- $R = \lim_{t\to\infty} R_t$: time average
- $\bar{R}(t) = \sum_{n=0}^{\infty} np_n(t)$**:** expected number of jobs at time t
- $\bar{R} = \lim_{t\to\infty} \bar{R}(t)$: steady-state expected number of jobs in the system

Ergodic System: $R = \bar{R}$

How to Computer R and $\bar{R}$?

Assumption: $E[X] < \infty$

Result: $R = \bar{R} = \frac{E[R_n]}{E[X]}$ (under some conditions)

Approach

- Renewal Process
- Renewal Reward Processes

Renewal Process

Assumptions: $X_1, X_2, ...$ positive IID r.v.;$\bar{X} < \infty$,

Results: (Note that $0 < \bar{X}$)

$lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\bar{X}}$

$lim_{t\to\infty} \frac{E[N(t)]}{t} = \frac{1}{\bar{X}}$

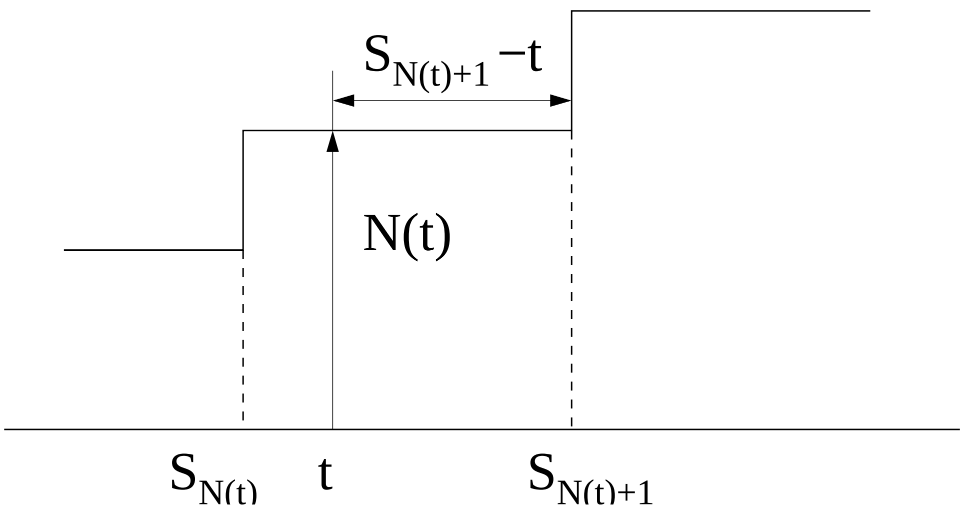
$lim_{t\to\infty} (E[N(t+\delta)] - E[N(t)]) = \frac{\delta}{\bar{X}}, \quad \delta > 0$

$m(t) = E[N(t)]$

Using Wald's equality:

- $m(t) \geq \frac{1}{E[X]} - \frac{1}{t}$
- $m(t) \leq \tilde{m}(t) \leq \frac{1}{E[\tilde{X}]} + \frac{b_t}{tE[\tilde{X}]} - \frac{1}{t}$,
  where we used

$$\{\tilde{m}(t); t > 0\}, \qquad b_t = \sqrt{t}.$$



$E[S_{N(t)+1} - t] = ???$

Blackwell's Theorem

Arithmetic Distribution: Renewals only happen at integer multiples of some real number $d > 0$.

Span of an Arithmetic Distribution: The largest real number d such that the above property holds

Blackwell's Theorem

- For non-arithmetic inter-renewal distribution and $\delta > 0$:

$$\lim_{t\to\infty} m(t+\delta) - m(t) = \delta \frac{1}{E[X]}$$

- For arithmetic inter-renewal distribution with span d and any **integer** $n \geq 1$

$$\lim_{t\to\infty} m(t+nd) - m(t) = nd \frac{1}{E[X]}$$

- For non-arithmetic inter-renewal distribution and $\delta > 0$:

$$\lim_{t\to\infty} P\Big(N(t+\delta) - N(t) = 0\Big) = 1 - \delta \frac{1}{E[X]} + o(\delta)$$

$$\lim_{t\to\infty} P\Big(N(t+\delta) - N(t) = 1\Big) = \delta \frac{1}{E[X]} + o(\delta)$$

$$\lim_{t\to\infty} P\Big(N(t+\delta) - N(t) \geq 2\Big) = o(\delta)$$

Note: Similar to Poisson process, but increments are **not necessarily independent**.

How to Computer R and $\bar{R}$?

Assumption: $E[X] < \infty$

Result: $R = \bar{R} = \frac{E[R_n]}{E[X]}$ (under some conditions)

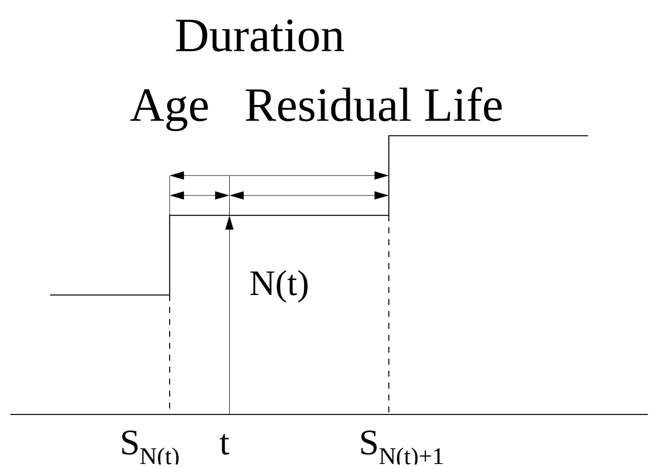
Approach

- Renewal Process
- Renewal Reward Processes

Renewal Reward Process

$\{N(t); t \geq 0\}$: Renewal Process
$\{R(t); t \geq 0\}$: Reward Function (need some assumptions)

- $R_n = \int_{S_{n-1}}^{S_n} R(t)dt$: IID r.v.
- $R = \bar{R} = \frac{E[R_n]}{E[X]}$ (non-arithmetic)



Renewal Reward Process - Results

- If $\bar{X} < \infty$ or $E[R_n] < \infty$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t R(\tau) d\tau = \frac{E[R_n]}{\bar{X}}, \qquad \text{with probability 1.}$$

- If non-arithmetic renewal process and $r(z)$ is directly Rieman integrable, then

$$\lim_{t\to\infty} E[R(t)] = \frac{E[R_n]}{\bar{X}}.$$

- If arithemtic renewal process with span d, then

$$\lim_{n\to\infty} E[R(nd)] = \frac{E[R_n]}{\bar{X}}.$$